# Finding real polynomials

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# Rabbits, Corabbits, and Airplanes

University of Michigan LoG(M)

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- Let f(z) be a polynomial such that $f(z) = z^2 + k$ for some $k \in \mathbb{C}$. Then $f'(x) = 0$ when $x = 0$.

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- f(0)
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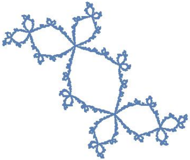
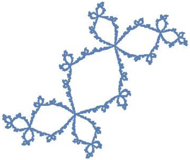
- f(f(f(0)))
- Rabbits, Corabbits, and Airplanes are special complex polynomials such that $f(f(f(0))) = 0$.
- These polynomials are called polynomials of cycle 3.



Rabbit

Airplane



Corabbit

# Motivation



- Our topic is blend of major topics, in particular complex dynamics, topology, and geometric group theory.
- We want to investigate what distortions we can make to our polynomials to make new and interesting ones.
- Our goal is to classify these changes using our three special polynomials: rabbit, airplane, and corabbit

# Homeomorphisms/Mapping Classes

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**Definition (Homeomorphism)**

**Homeomorphisms** are continuous bijective maps from $\mathbb{C} \to \mathbb{C}$ with continuous inverses.

- Essentially, homeomorphisms are analogous to isomorphisms.

# Mapping Classes and Polynomials

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- Mapping classes are classes of homeomorphisms that are "similar" enough that we can wiggle from homeomorphism to another
- They are easier to study than homeomorphisms, which there is an infinite amount

# Mapping Classes and Dehn Twists

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## Definition (Dehn Twist)

**Dehn twists** are twists along a particular curve.

## Theorem (Dehn)

*Let $\phi$ be a finite mapping class. Then $\phi$ can be written as the product of finite Dehn twists.*

- In particular, we want to write f as a product of Dehn Twists along curves x, y, and z.

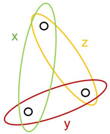


**Figure:** The curves enclosing the points in the post-critical set

# Lifting



**Definition (Lifting)**

Let $\phi : \mathbb{C} \mapsto \mathbb{C}$ be a homeomorphism. $\phi$ is said to **lift** under f if there exists $\widetilde{\phi}$ such that f $\circ\, \widetilde{\phi} = \phi \circ$ f.

- In practical terms, a homeomorphism lifts under f if we can trace from a marked point along its curve without bumping into other marked points.
- In particular, we are interested in the homeomorphism f represented by the composition of a stretch of the complex plane and a counterclockwise rotation.

# How Do We Know if it Lifts?

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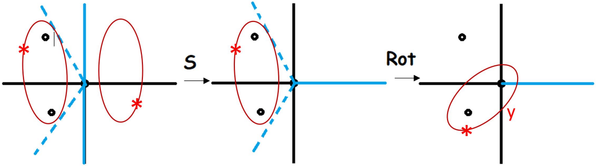


**Figure:** Transformation of curve y under f

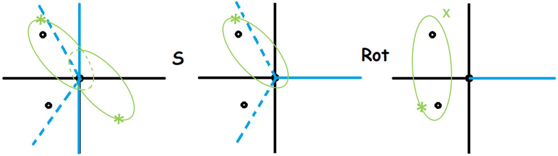


**Figure:** Transformation of curve x under f

# Rules for Lifting



- If the sum of the exponents on $T_x$ is even, then the product of twists will lift.
- Certain "chunks" lift to other mappings. In particular:
  - $T_x{}^2$ lifts to $T_x{}^{-1}T_y{}^{-1}$
  - $T_y$ lifts to $T_x$
  - $T_xT_yT_x{}^{-1}$ lifts to the identity
- Through these rules, we can also establish other rules using rules for inverses. For example, $T_x{}^{-2}$ lifts to $(T_x{}^{-1}T_y{}^{-1})^{-1}$ or $T_yT_x$
- We stop lifting when we reach one of three end conditions:
  - Id: Rabbit
  - $T_x$: Airplane
  - $T_x^{-1}$: Corabbit

# Lifting Example 1



**If the product lifts,**
$\mathrm{T}_x \mathrm{T}_y^2 \mathrm{T}_x^{-1} =$

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Multiply by $\mathrm{Id} = (T_x^{-1} T_x)$ :

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$(T_x T_y T_x^{-1})(T_x T_y T_x^{-1}) =$

Apply lifting rules:

$\mathrm{Id} \cdot \mathrm{Id} =$

Id

End condition, we stop

# Lifting Example 2

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**If the product does not lift,**

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1. Multiply $T_x^{-1}$ on the left

# Lifting Example 2



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**If the product does not lift,**

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$T_y T_x =$

1. Multiply $T_x^{-1}$ on the left

$T_x^{-1} T_y T_x =$

2. Apply the lifting rules

Id

3. Multiply $T_x$ on the right

$T_x$

End condition, we stop

# Methods



We can use code to automate this algorithmic process.

1. Generate permutations of length n
2. For each permutation, determine if it lifts.
3. Depending on whether if lifts, follow the steps demonstrated previously.
4. Once we reach an end condition, classify the twisted polynomial as rabbit, corabbit, or airplane.

# Goals



- For now, we plan on determining which insertions are most effective to rewrite the product as chunks.
  - One possible method is by iterating in pairs through a permutation and inserting $T_x^{-1}T_x$ or moving on to the next pair.
- We plan to use the rules explained on the previous slide to determine the percentage of polynomials that are either rabbit, corabbit, or airplane.

# References



Lab of Geometry at Michigan. LoG(M) Beamer Template. *University of Michigan Department of Mathematics*. 2018.

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