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SET Marble-Run Computer

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LOG(M)

Introduction

Goals and Set Up

1. Understanding Finite Fields and Fourier Transform in terms of counting SETs.
2. Using mechanical computation to count the number of SETs in a collection of cards.

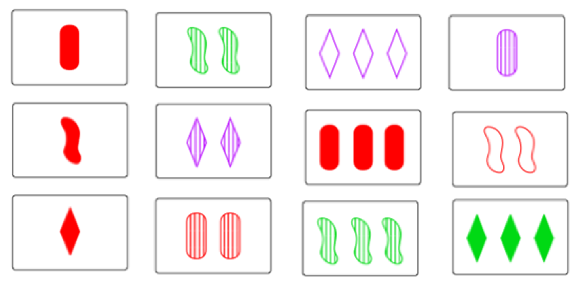
Our final product takes the form of a human dropping marbles through a marble run that represents the face-up cards then letting loose the marbles. The final count of marbles in a bin reveals the number of SETs among face up cards.

1. The Game SET

1.1 Rules

A SET deck has 81 cards. Each card has one, two, or three symbols, with a shape, shading, and color chosen from a collection of three values. The possible attributes are:

Number: One, Two, Three
Shading: Solid, Striped, Open
Color: Red, Green, Purple
Shape: Ovals, Squiggles, Diamonds



The goal of this game is to find tricks of three cards (called a SET) in which, for each attribute, the cards all have the same value of the attribute or have all three values.

1.2 Defining Cards and Counting SETS

Definition Let F_3 be the field with three elements, and consider the vector space F_3^4. A point of F_3^4 is a 4-tuple of the form (x_1, x_2, x_3, x_4), such that $x_1, x_2, x_3, x_4 \in \{0, 1, 2\}$.

x_1 (Shape)	x_2 (Shading)	x_3 (Color)	x_4 (Number)
0: Oval	0: Open	0: Red	0: Three Shapes
1: Diamonds	1: Striped	1: Green	1: One Shape
2: Squiggle	2: Solid	2: Purple	2: Two Shapes

Example 1. A card with coordinate (1, 2, 1, 2) is equivalent to the card with Two Solid Green Diamonds.

Number of Non-zero Vectors: $3^4 - 1 = 80$
Number of Directions: $\frac{1}{3-1} \cdot (3^4 - 1) = 40$
Number of Lines in each direction: 3^3
Number of Sets: $40 \cdot (3^4) = 1080$

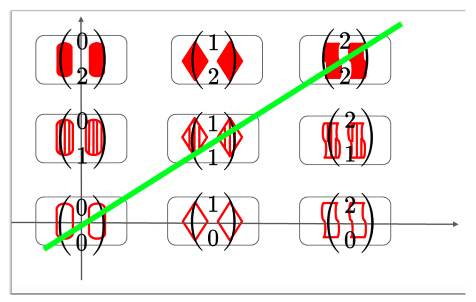


Illustration of counting the total number of SETS in F_3^2

2. The Fourier Transform

Definition Given a Function $f : \mathbb{F}_3^d \to \mathbb{C}$, define the Fourier Transform of f to be a new function $\hat{f} : \mathbb{F}_3^d \to \mathbb{C}$ defined by the formula:

$$\hat{f}(x) = \sum_{x \in \mathbb{F}_3^d} f(x)\omega^{z \cdot x}$$

where $\omega = e^{\frac{2\pi i}{3}}$. Given a set $S \subset \mathbb{F}_3^d$, the *characteristic function of S* is defined by the formula:

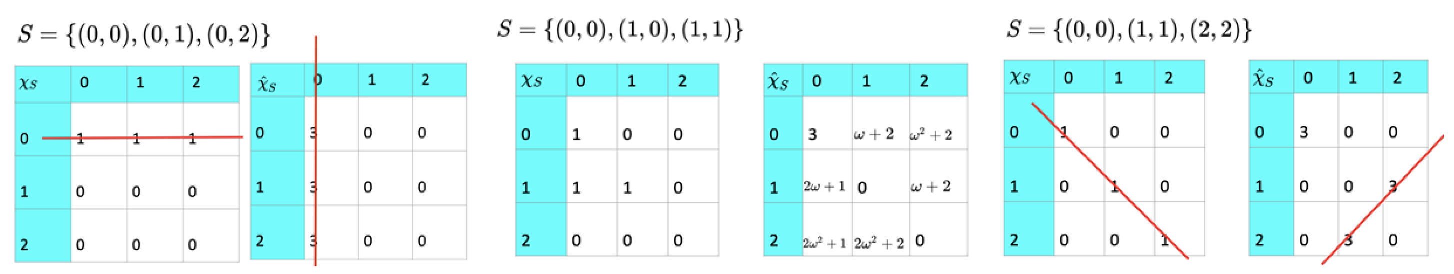
$$\chi(x) = \begin{cases} 1 & if\ x \in S \\ 0 & if\ x \notin S \end{cases}$$

Example 2.(Fourier Transform for F_3^2)

- Take subset S = {(0,0), (1,0), (1,1)}

- $\hat{\chi}((0,0)) = \omega^{(0,0)(0,0)} + \omega^{(1,0)(0,0)} + \omega^{(1,1)(0,0)} = 3$
- $\hat{\chi}((0,1)) = \omega^{(0,0)(0,1)} + \omega^{(1,0)(0,1)} + \omega^{(1,1)(0,1)} = \omega + 2$
- $\hat{\chi}((0,2)) = \omega^{(0,0)(0,2)} + \omega^{(1,0)(0,2)} + \omega^{(1,1)(0,2)} = \omega^2 + 2$
- $\hat{\chi}((1,0)) = \omega^{(0,0)(1,0)} + \omega^{(1,0)(1,0)} + \omega^{(1,1)(1,0)} = 2\omega + 1$
- $\hat{\chi}((1,1)) = \omega^{(0,0)(1,1)} + \omega^{(1,0)(1,1)} + \omega^{(1,1)(1,1)} = 0$
- $\hat{\chi}((1,2)) = \omega^{(0,0)(1,2)} + \omega^{(1,0)(1,2)} + \omega^{(1,1)(1,2)} = \omega + 2$
- $\hat{\chi}((2,0)) = \omega^{(0,0)(2,0)} + \omega^{(1,0)(2,0)} + \omega^{(1,1)(2,0)} = 2\omega^2 + 1$
- $\hat{\chi}((2,1)) = \omega^{(0,0)(2,1)} + \omega^{(1,0)(2,1)} + \omega^{(1,1)(2,1)} = 2\omega^2 + 2$
- $\hat{\chi}((2,2)) = \omega^{(0,0)(2,2)} + \omega^{(1,0)(2,2)} + \omega^{(1,1)(2,2)} = 0$

Follow this calculation, finally we will get the following Fourier table, indicating a certain pattern:



Note: We notice that in case 1 and 3, the subset S can form a SET, and we get two lines that are perpendicular to each other. However, since the set S in case 2 cannot form a set, we did not find such pattern. Also notice that as we are working in mod 3, vector $\langle 0,0 \rangle$ is equivalent to vector $\langle 0,3 \rangle$.

Proposition 2. In the context of the game SET, points represent the cards and lines represent the SETs. Let S be a subset of F_3^d that contains p points and l lines. Then $p + 6l = \frac{1}{3^d} \sum_{x \in \mathbb{F}_3^d} \tau(z)^3$ where $\tau(z)$ is the hyperplane triple function of S.

Example 3.

χ	0	1	2
0	1	0	0
1	1	0	0
2	1	0	0

$\hat{\chi}$	0	1	2
0	3	3	3
1	0	0	0
2	0	0	0

$\hat{\chi}^3$	0	1	2
0	27	27	27
1	0	0	0
2	0	0	0

$LHS = p + 6l = 3 + 6 * 1 = 9 = RHS = \frac{1}{3^2} \sum (\hat{\chi})^3 = \frac{1}{9}(27 + 27 + 27) = 9$
$p = 3$ because we have 3 cards, and $l = 1$ because the cards form a line.

Proposition 3. $\hat{\chi}_{AS+\vec{V}}(\vec{X}) = \hat{\chi}_{AS}(\vec{X})\omega^{\vec{X}\cdot\vec{V}} = \hat{\chi}_S(A^T\vec{X})\omega^{\vec{X}\cdot\vec{V}}$

By Proposition 3, we can perform linear and non-linear transformation on the subset S to extend the Fourier Transform to different sets of cards.

3. Methods and Results

3.1 Building A Physical Calculator Using Straws

The following illustration considers a nine card game, with each card having one, two or three symbols. The symbols are denoted as a bullet, star and circle. Each card gets a vector of nine numbers mod 7, where arithmetic mod 7 replaces $\mathbb{C}$ in the Fourier transform and 2 replaces $e^{\frac{2\pi i}{3}}$ as a cube root of unity.

Fourier Vectors of the 9-cards (Fourier Transformation under mod 7).

○○○	1	1	1	1	1	1	1	1	1
●●●	1	4	4	1	4	2	2	1	2
●	1	4	1	4	2	2	1	2	4
○○	1	1	4	2	2	1	2	4	4
●●	1	4	2	2	1	2	4	4	1
⋆⋆⋆	1	2	2	1	2	4	4	1	4
⋆⋆	1	2	1	2	4	4	1	4	2
○	1	1	2	4	4	1	4	2	2
⋆	1	2	4	4	1	4	2	2	1

Since the left column and the top row always return 1, we will ignore these two lines. We then notice that the Fourier Transform of the remaining 16 columns follows a cyclic nature. The Fourier Transform is read off the nine columns indicated by arrows below.

read:	↓	↓		↓		↓		↓		↓		↓		↓		↓	
○○○	1	1	4	1	4	1	1	1	4	1	2	1	2	1	1	1	2
●●●	1	4	1	4	1	1	1	4	1	2	1	2	1	1	1	2	1
●	1	4	1	1	1	4	1	2	1	2	1	1	1	2	1	4	1
○○	1	1	1	4	1	2	1	2	1	1	1	2	1	4	1	4	1
●●	1	4	1	2	1	2	1	1	1	2	1	4	1	4	1	1	1
⋆⋆⋆	1	2	1	2	1	1	1	2	1	4	1	4	1	1	1	4	1
⋆⋆	1	2	1	1	1	2	1	4	1	4	1	1	1	4	1	2	1
○	1	1	1	2	1	4	1	4	1	1	1	4	1	2	1	2	1
⋆	1	2	1	4	1	4	1	1	1	4	1	2	1	2	1	1	1
read:	↑	↑		↑		↑		↑		↑		↑		↑		↑	

- **Algorithm of Using the Marble-Run Calculator**:
  1. To get the Fourier Vector of S, we ignore the first number (i.e, 1), and we start from the first card.
  2. Rotate the octagon calculator so that it represents the Fourier Vector of your second card. Then match the octagon calculator with the Fourier Vector of your first card (i.e, find the corresponding cyclic nature).
  3. Write down the answer you get.
  4. Repeat step 2 with the answer you got in step 3 and match it with the third card.
  5. Repeat step 4 until there is no card left.
- Cube and Sum: Now you get a new Fourier Vector of 8 dimensions. Since we ignored the first column when doing the cyclic calculation, we add $1 * |S|$, where $|S|$ is the number of cards you have, at the beginning to get a 9-dimensional Fourier Vector. We take the cube of all the numbers in our Fourier Vector and sum them up.
- Division: By our Proposition 2, we divide the summation from the last step by 3^d ($d = 2$ in our example, since we are working with 9 cards with 2 characteristics. So $3^d = 3^2 = 9 \bmod 7 = 2$).
- By proposition 2, the number of SET we have in S is equal to $|S|$ minus the number we got from our last step. Notice that $p + 6l = p - l$ under mod 7 arithmetic.