SET Marble-Run Computer

Lingcong Xu, Amanda Yao, Yi Zhou
Faculty Advisor: Martin Strauss
Graduate Mentor: Salman Siddiqi, Lizbee Collins-Wildman

26 February, 2019

Motivation and Overview

Lingcong Xu, Amanda Yao, Yi Zhou
Faculty Advisor: Martin Strauss
Graduate Mentor: Salman Siddiqi, Lizbee Collins-Wildman

Motivation

1. Understanding Finite Fields and Fourier Transform in terms of counting sets.

2. Using mechanical computation to count the number of sets in a collection of cards.

Overview

1. SET Game

2. Finite Fields

3. Application of Fourier Transform

SET Game

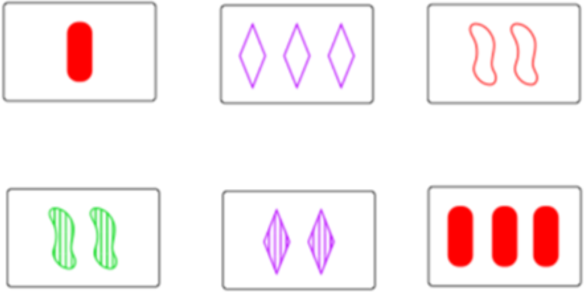
Lingcong Xu, Amanda Yao, Yi Zhou
Faculty Advisor: Martin Strauss
Graduate Mentor: Salman Siddiqi, Lizbee Collins-Wildman

A SET deck has 81 cards, one card for each of the possible values of the following possible attributes:

Number: One, Two, Three
Shading: Solid, Striped, Open
Color: Red, Green, Purple
Shape: Ovals, Squiggles, Diamonds

Goal:
Find 3 cards that are either all the same or all different with respect to each of the four attributes.

SET Game



Finite Fields

Let F_3 be the field with three elements.
A point in vector space F_3^2 is a 2-tuple of the form (x, y) such that

$$x, y \in \{0, 1, 2\}$$

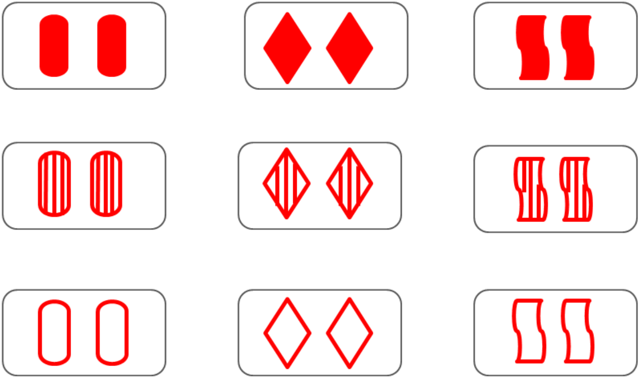
X (Shape)	Y (Shading)
0 = Oval	0 = Open
1 = Diamonds	1 = Striped
2 = Squiggles	2 = Solid

3 ways to count SETs

1. Combinatorics
2. Geometry (from lines to vectors)
3. Consider the cards themselves

counting SETs

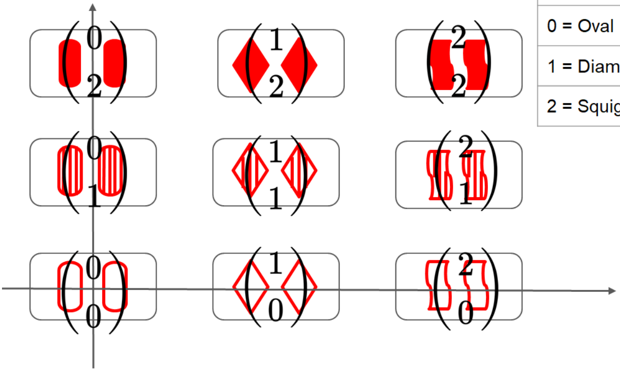
Lingcong Xu, Amanda Yao, Yi Zhou
Faculty Advisor: Martin Strauss
Graduate Mentor: Salman Siddiqi, Lizbee Collins-Wildman



counting SETs

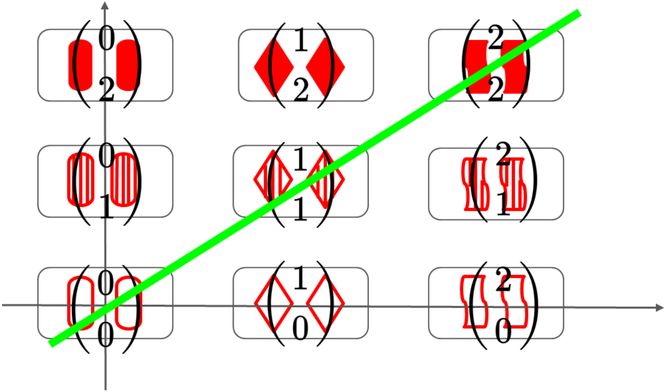
SET Marble-Run Computer

Lingcong Xu, Amanda Yao, Yi Zhou
Faculty Advisor: Martin Strauss
Graduate Mentor: Salman Siddiqi, Lizbee Collins-Wildman



X (Shape)	Y (Shading)
0 = Oval	0 = Open
1 = Diamonds	1 = Stripped
2 = Squiggles	2 = Solid

counting SETs



12 SETs

Z_3^2

- Vectors: $3^2 - 1 = 8$
- Directions: $\frac{1}{2} \cdot (3^2 - 1) = 4$
- Lines in each direction: 3
- Sets: $4 \cdot (3) = 12$

Z_3^4

- Vectors: $3^4 - 1 = 80$
- Directions: $\frac{1}{2} \cdot (3^4 - 1) = 40$
- Lines in each direction: 3^3
- Sets: $40 \cdot (3^3) = 1080$

Z_3^d

- Vectors: $3^d - 1$
- Directions: $\frac{1}{2} \cdot (3^d - 1)$
- Lines in each direction: 3^{d-1}
- Sets: $\frac{1}{2} \cdot (3^d - 1) \cdot 3^{d-1}$

Consider Cards

Total number of SETS:

$$\frac{\binom{81}{2}}{\binom{3}{2}}$$

Fourier Transform Definition

Given a Function $f : \mathbb{F}_3^d \to \mathbb{C}$, define the *Fourier transform* of f to be a new function $\hat{f} : \mathbb{F}_3^d \to \mathbb{C}$ defined by the formula:

$$\hat{f}(x) = \sum_{x \in \mathbb{F}_3^d} \chi(x) \omega^{z*x}$$

where $\omega = e^{\frac{2\pi i}{3}}$. Given a set $S \subset \mathbb{F}_3^d$, the *characteristic function of S* is defined by the formula:

$$\chi(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Fourier Transform for F_3^2

Lingcong Xu, Amanda Yao, Yi Zhou
Faculty Advisor: Martin Strauss
Graduate Mentor: Salman Siddiqi, Lizbee Collins-Wildman

S = {(0,0), (0,1), (0,2)}

$$\hat{X}((0,0)) = \omega^{(0,0)(0,0)} + \omega^{(0,1)(0,0)} + \omega^{(0,2)(0,0)} = 3$$
$$\hat{X}((0,1)) = \omega^{(0,0)(0,1)} + \omega^{(0,1)(0,1)} + \omega^{(0,2)(0,1)} = 0$$
$$\hat{X}((0,2)) = \omega^{(0,0)(0,2)} + \omega^{(0,1)(0,2)} + \omega^{(0,2)(0,2)} = 0$$
$$\hat{X}((1,0)) = \omega^{(0,0)(1,0)} + \omega^{(0,1)(1,0)} + \omega^{(0,2)(1,0)} = 3$$
$$\hat{X}((1,1)) = \omega^{(0,0)(1,1)} + \omega^{(0,1)(1,1)} + \omega^{(0,2)(1,1)} = 0$$
$$\hat{X}((1,2)) = \omega^{(0,0)(1,2)} + \omega^{(0,1)(1,2)} + \omega^{(0,2)(1,2)} = 0$$
$$\hat{X}((2,0)) = \omega^{(0,0)(2,0)} + \omega^{(0,1)(2,0)} + \omega^{(0,2)(2,0)} = 3$$
$$\hat{X}((2,1)) = \omega^{(0,0)(2,1)} + \omega^{(0,1)(2,1)} + \omega^{(0,2)(2,1)} = 0$$
$$\hat{X}((2,2)) = \omega^{(0,0)(2,2)} + \omega^{(0,1)(2,2)} + \omega^{(0,2)(2,2)} = 0$$

Fourier Transform for F_3^2

Lingcong Xu, Amanda Yao, Yi Zhou
Faculty Advisor: Martin Strauss
Graduate Mentor: Salman Siddiqi, Lizbee Collins-Wildman

S = {(0,0), (1,0), (1,1)}

$$\hat{X}((0,0)) = \omega^{(0,0)(0,0)} + \omega^{(1,0)(0,0)} + \omega^{(1,1)(0,0)} = 3$$
$$\hat{X}((0,1)) = \omega^{(0,0)(0,1)} + \omega^{(1,0)(0,1)} + \omega^{(1,1)(0,1)} = \omega + 2$$
$$\hat{X}((0,2)) = \omega^{(0,0)(0,2)} + \omega^{(1,0)(0,2)} + \omega^{(1,1)(0,2)} = \omega^2 + 2$$
$$\hat{X}((1,0)) = \omega^{(0,0)(1,0)} + \omega^{(1,0)(1,0)} + \omega^{(1,1)(1,0)} = 2\omega + 1$$
$$\hat{X}((1,1)) = \omega^{(0,0)(1,1)} + \omega^{(1,0)(1,1)} + \omega^{(1,1)(1,1)} = 0$$
$$\hat{X}((1,2)) = \omega^{(0,0)(1,2)} + \omega^{(1,0)(1,2)} + \omega^{(1,1)(1,2)} = \omega + 2$$
$$\hat{X}((2,0)) = \omega^{(0,0)(2,0)} + \omega^{(1,0)(2,0)} + \omega^{(1,1)(2,0)} = 2\omega^2 + 1$$
$$\hat{X}((2,1)) = \omega^{(0,0)(2,1)} + \omega^{(1,0)(2,1)} + \omega^{(1,1)(2,1)} = 2\omega^2 + 2$$
$$\hat{X}((2,2)) = \omega^{(0,0)(2,2)} + \omega^{(1,0)(2,2)} + \omega^{(1,1)(2,2)} = 0$$

Pattern

$$S = \{(0,0), (0,1), (0,2)\}$$

χ_S	0	1	2
0	1	1	1
1	0	0	0
2	0	0	0

$\hat{\chi}_S$	0	1	2
0	3	0	0
1	3	0	0
2	3	0	0

Pattern

$$S = \{(0,0),(1,1),(2,2)\}$$

χ_S	0	1	2
0	1	0	0
1	0	1	0
2	0	0	1

$\hat{\chi}_S$	0	1	2
0	3	0	0
1	0	0	3
2	0	3	0

Future Goals

1. Understanding the pattern of the Fourier Transform

2. Developing algorithm for mechanical computation to count the number of sets in a collection of cards.

References

[1] Davis Benjamin and Maclagan Diane, *The Card Game Set*,print, https://homepages.warwick.ac.uk/staff/D.Maclagan/papers/set.pdf