Due date: 01/25. Please hand in your work at the beginning of the class.

**Exercise 1:** Derive the orthogonality relations

\[
\int_{-\pi}^{\pi} \cos m\theta \cos n\theta d\theta = \begin{cases} 
0 & m \neq n \\
\pi & m = n 
eq 0 \\
2\pi & m = n = 0 
\end{cases} \quad \text{for all } m = 0, 1, 2 \ldots, n = 0, 1, 2, \ldots ,
\]

\[
\int_{-\pi}^{\pi} \sin m\theta \sin n\theta d\theta = \begin{cases} 
0 & m \neq n \\
\pi & m = n 
\end{cases} \quad \text{for all } m = 1, 2 \ldots, n = 1, 2, \ldots .
\]

\[
\int_{-\pi}^{\pi} \cos m\theta \sin n\theta d\theta = 0, \quad \text{for all } m = 0, 1, 2 \ldots, n = 1, 2, \ldots .
\]

Work out the Fourier series of \( f \) in Exercise 2–3, given over one period as follows.

**Exercise 2:**

\[
f(\theta) = \begin{cases} 
-1 & -\pi < \theta < 0 \\
1 & 0 < \theta \leq \pi \\
0 & \theta = 0 
\end{cases}
\]

**Exercise 3:**

\[
f(\theta) = e^{-\theta}, \quad 0 < \theta \leq 2\pi.
\]

**Exercise 4:** Plot a few partial sums of the Fourier series obtain in Exercise 2 and 3, what can you say about the convergence of the Fourier series to the original function at fixed \( \theta \in \mathbb{R} \)?

**Exercise 5:** Suppose \( f \) is 2\( \pi \)-periodic and \( \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta < \infty \). The Fourier series of \( f \) is given by

\[
a_0/2 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).
\]
Prove the Bessel’s inequality (do not use the Bessel’s inequality in terms of $c_n$)

\[
\frac{1}{4}a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta.
\]

*Hint:* You may use Exercise 1.