Homework 10 solution

Problem 1: (a) Evaluate the line integral \( \int_{\gamma_1} z \, dz \) where \( \gamma_1 \) is the straight line from \( z = 0 \) to \( z = 1 + i \). (b) After that evaluate \( \int_{\gamma_2} e^z \, dz \) where \( \gamma_2 \) is a union of two straight lines, one from \( z = i \) to \( z = 1 + i \), the second from \( z = 1 + i \) to \( z = 1 - 2i \). (c) Consider the line integral \( \int_{\gamma} z \, dz \) for any path \( \gamma \) starting at \( z = 0 \) and terminating at \( z = 1 + i \). Show that this integral is path dependent by choosing two different paths from 0 to 1+i and obtaining different values for the line integral.

Solution: The straight line from \( z = 0 \) to \( z = 1 + i \) can be parametrized as \( \gamma(t) = t(1 + i), t \in [0, 1] \), then

\[
\int_{\gamma_1} z \, dz = \int_0^1 \frac{t}{t(1+i)}(1+i) \, dt = 2 \int_0^1 t \, dt = 1.
\]

For the second integral we use \( \gamma_{21}(t) = (1 - t)i + (1 + i)t, t \in [0, 1] \) for the first line segment from \( z = i \) to \( z = 1 + i \) and \( \gamma_{22}(t) = (1 - t)(1 + i) + (1 - 2i)t, t \in [0, 1] \) for the second from \( z = 1 + i \) to \( z = 1 - 2i \). Then

\[
\int_{\gamma_2} e^z \, dz = \int_{\gamma_{21}} e^z \, dz + \int_{\gamma_{22}} e^z \, dz = \int_0^1 e^{(1-t)i+(1+i)t} \, dt + \int_0^1 e^{(1-t)(1+i)+(1-2i)t} (-3i) \, dt
\]

\[
= e^{(1-t)i+(1+i)t} \bigg|_{t=0}^1 + e^{(1-t)(1+i)+(1-2i)t} \bigg|_{t=0}^1 = e^{1-2i} - e^i.
\]

Choose first \( \gamma_1(t) = t(1 + i), t \in [0, 1] \) and find

\[
\int_{\gamma_1} z \, dz = 1.
\]

Second pick

\[
\gamma_2(t) = \begin{cases} 
  t, & t \in [0, 1] \\
  1 + i(t - 1), & t \in [1, 2]
\end{cases}
\]
which parametrizes the union of two straight line segments: first from $z = 0$ to $z = 1$, after that from $z = 1$ to $z = 1 + i$. We find

$$\int_{\gamma_2} z \, dz = \int_0^1 t \, dt + \int_1^2 (1 - i(t - 1))i \, dt = 1 + i \neq 1,$$

i.e. the line integral is indeed dependent of path.

**Problem 2:** Evaluate each of the following integrals.

(a) \[ \int_{|z| = 1} \Im z \, dz \]

(b) \[ \int_{|z| = 1} \frac{dz}{z(z + 2)} \]

(c) \[ \int_{|z| = 3} \frac{z}{z^3 - 1} \, dz \]

**Solution:** In (a) we choose the parametrization $\gamma(t) = e^{it}, t \in [0, 2\pi)$ and find

$$\int_{|z| = 1} \Im z \, dz = \int_0^{2\pi} \sin t \, (ie^{it}) \, dt = \frac{1}{2} \int_0^{2\pi} (e^{2it} - 1) \, dt = -\pi.$$

For (b) use partial fractions

$$\int_{|z| = 1} \frac{dz}{z(z + 2)} = \frac{1}{2} \int_{|z| = 1} \left( \frac{1}{z} - \frac{1}{z + 2} \right) \, dz = i\pi$$

since $\int_{|z| = 1} \frac{dz}{z} = 2\pi i$ by the example we did in class or the Cauchy’s integral formula and $\int_{|z| = 1} \frac{dz}{z + 2} = 0$ by Cauchy’s theorem. Also in (c) we use partial fractions

$$\frac{z}{z^3 - 1} = \frac{1}{3} \left( \frac{1}{z - 1} + \frac{e^{i\frac{2\pi}{3}}}{z - e^{i\frac{2\pi}{3}}} + \frac{e^{i\frac{4\pi}{3}}}{z - e^{i\frac{4\pi}{3}}} \right)$$
so that

\[
\int \frac{z}{z^3 - 1} \, dz = \frac{1}{3} \int \frac{dz}{z} + \frac{1}{3} e^{i\frac{2\pi}{3}} \int \frac{dz}{z - e^{i\frac{2\pi}{3}}} + \frac{1}{3} e^{i\frac{4\pi}{3}} \int \frac{dz}{z - e^{i\frac{4\pi}{3}}}
\]

\[
= 2\pi i \left( \frac{1}{3} + \frac{1}{3} e^{i\frac{2\pi}{3}} + \frac{1}{3} e^{i\frac{4\pi}{3}} \right) = 0
\]

**Problem 3:** Evaluate each integral using the (generalized) Cauchy integral formula

(a) \( \int \frac{\sin z}{z} \, dz \)

(b) \( \int \frac{z + 1}{(z - 1)(z + 2)^3} \, dz \)

(c) \( \int \frac{z^2 - 1}{z^2 + 1} e^z \, dz \)

(d) \( \int \frac{z^3}{z^2 + 1} \, dz \)

where all the contours are oriented counterclockwise.

**Solution:** Apply the Cauchy integral formula in (a) to obtain that

\[
\int \frac{\sin z}{z} \, dz = 2\pi i \sin(0) = 0.
\]

For (b), either use partial fractions or a contour deformation argument (i.e. introduce two small circles of radius \( r \in (0, \frac{3}{2}) \) around \( z = -2 \) and \( z = 1 \)) leading to

\[
\int \frac{z + 1}{(z - 1)(z + 2)^3} \, dz = \int \frac{z + 1}{z - 1} \left( \frac{1}{(z + 2)^3} \right) \, dz + \int \frac{z + 1}{z - 1} \left( \frac{1}{z + 2} \right) \, dz
\]

\[
= \frac{2\pi i}{2!} \frac{d^2}{dz^2} \left. \left( \frac{z + 1}{z - 1} \right) \right|_{z=-2} + 2\pi i \left. \frac{z + 1}{(z + 2)^3} \right|_{z=1}
\]

\[
= i\pi \left( -\frac{4}{27} \right) + 2\pi i \left( \frac{2}{27} \right) = 0.
\]
Also in part (c) we use a contour deformation argument and introduce two small circles of radius \( r \in (0, 1) \) around \( z = \pm i \):

\[
\int |z|=3 \frac{z^2 - 1}{z^2 + 1} e^z \, dz = \int_{|z-i|=r} \frac{z^2-1}{z} e^z \, dz + \int_{|z+i|=r} \frac{z^2-1}{z} e^z \, dz
\]

\[
= 2\pi i \frac{z^2 - 1}{z + i} \bigg|_{z=i} + 2\pi i \frac{z^2 - 1}{z - 1} \bigg|_{z=-i} = -4\pi i \sin(1).
\]

Finally with yet another contour deformation (introducing two small circles of radius \( r \in (0, 1) \) around \( z = e^{-i\pi/4} \) and \( z = e^{i3\pi/4} \)) we find

\[
\int_{|z|=3} \frac{z^3}{z^2 + 1} \, dz = \int_{|z-e^{-i\pi/4}|=r} \frac{z^3}{z - e^{-i\pi/4}} \, dz + \int_{|z-e^{i3\pi/4}|=r} \frac{z^3}{z - e^{i3\pi/4}} \, dz
\]

\[
= 2\pi i \frac{z^3}{z - e^{i3\pi/4}} \bigg|_{z=e^{-i\pi/4}} + 2\pi i \frac{z^3}{z - e^{-i\pi/4}} \bigg|_{z=e^{i3\pi/4}} = 2\pi.
\]

**Problem 4:** Evaluate the given integral by means of the residue theorem.

(a) \( \int_{|z|=1} \frac{dz}{z^2 e^z} \)  
(b) \( \int_{|z|=1} \frac{z^2 \, dz}{\sinh(2z)} \)  
(c) \( \int_{|z-1|=2} \frac{dz}{z^2 - 2iz - 2'} \)

where all the contours are oriented counterclockwise.

**Solution:** The integrand \( f(z) = \frac{1}{z^2 e^z} = \frac{e^{-z}}{z^2} \) in (a) has a second order pole singularity at \( z = 0 \) since

\[
f(z) = \frac{e^{-z}}{z^2} = \frac{1}{z^2} \left(1 - z + \cdots \right), \quad z \to 0
\]

Thus \( \text{Res}(f; 0) = -1 \) and \( z_0 = 0 \) is in fact the only isolated singularity of the
integrand. Hence, by residue theorem

\[ \int_{|z|=1} \frac{dz}{z^2 e^z} = -2\pi i. \]

The integrand \( g(z) = \frac{z^2}{\sinh(2z)} \) in (b) has isolated singularities at \( z = z_k = \frac{i\pi}{2} k, k \in \mathbb{Z}, \)

but only \( z_0 = 0 \) is contained within \( |z| = 1. \) Since \( \sinh(2z) = 2z + \cdots, z \to z_0 \) has a zero of order 1, and \( z^2 \) has a zero of order 2, then \( z = 0 \) is a removable singularity and we see that

\[ \int_{|z|=1} \frac{z^2}{\sinh(2z)} \, dz = 0. \]

The integrand \( h(z) = \frac{1}{z^2 - 2iz - 2} = \frac{1}{(z - 1 - i)(z - 1 + i)} \) has two first order poles at \( z = \pm 1 + i \)

but only \( z_1 = 1 + i \) is within \( |z - 1| = 2, \) since \( \text{Res}(h; 1 + i) = \frac{1}{2} \) then

\[ \int_{|z-1|=2} \frac{dz}{z^2 - 2iz - 2} = i\pi. \]

**Problem 5:** Evaluate the given integral by means of the residue theorem.

(a) \( \int_0^{2\pi} \frac{dt}{(a + b \cos t)^2}, \) \( a > b > 0 \) \hspace{1cm} (b) \( \int_0^{\pi} \frac{dt}{(a + \cos t)^2}, \) \( a > 1. \)

**Solution:** We apply the change of variables \( z = e^{it}. \) In part (a)

\[ \int_0^{2\pi} \frac{dt}{(a + b \cos t)^2} = \int_{|z|=1} \frac{1}{iz (a + \frac{b}{2}(z + z^{-1}))^2} \, dz. \]

Now we have

\[ f(z) = \frac{1}{iz (a + \frac{b}{2}(z + z^{-1}))^2} = \frac{4}{ib^2} \frac{z}{(z - z_1)^2(z - z_2)^2} \]
with \( z_1 = \frac{1}{b}(\sqrt{a^2 - b^2} - a) \) and \( z_2 = -\frac{1}{b}(\sqrt{a^2 - b^2} + a) \). However only \( z_1 \) is within \( |z| = 1 \) since \( z_1z_2 = 1 \) and \( |z_2| > 1 \) in view of \( a > b > 0 \). Thus, by residue theorem

\[
\int_0^{2\pi} \frac{dt}{(a + b \cos t)^2} = 2\pi i \text{Res}(f; z_1)
\]

and

\[
\text{Res}(f; z_1) = \frac{1}{2!} \frac{d}{dz} \left[ f(z)(z - z_1)^2 \right] = \frac{4}{ib^2(z_1 - z_2)^2} \left( \frac{1}{z_1} - \frac{2}{z_1 - z_2} \right).
\]

Together,

\[
\int_0^{2\pi} \frac{dt}{(a + b \cos t)^2} = 2\pi i \frac{4}{ib^2(z_1 - z_2)^2} \left( \frac{1}{z_1} - \frac{2}{z_1 - z_2} \right) = \frac{2\pi a}{(a^2 - b^2)^{3/2}}
\]

since \( z_1 + z_2 = -\frac{2a}{b}, z_1z_2 = \frac{2}{b}\sqrt{a^2 - b^2} \). In part (b) we note first that

\[
\int_0^{\pi} \frac{dt}{(a + \cos t)^2} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{dt}{(a + \cos t)^2}
\]

since the integrand is an even function. Then

\[
\int_{-\pi}^{\pi} \frac{dt}{(a + \cos t)^2} = \int_{|z| = 1} \frac{1}{i\pi} \frac{1}{(a + \frac{1}{2}(z + z^{-1}))^2} dz.
\]

Now we have

\[
f(z) = \frac{1}{i\pi} \frac{1}{(a + \frac{1}{2}(z + z^{-1}))^2} = \frac{4}{i} \frac{z}{(z - z_3)^2(z - z_4)^2}
\]

with \( z_3 = -a + \sqrt{a^2 - 1} \) and \( z_4 = -a - \sqrt{a^2 - 1} \). But only \( z_3 \) is within \( |z| = 1 \) in view of \( a > 1 \) similarly

\[
\int_0^{\pi} \frac{dt}{(a + \cos t)^2} = 2\pi i \frac{2}{i(z_3 - z_4)} \left( \frac{1}{z_3} - \frac{2}{z_3 - z_4} \right) = \frac{\pi a}{(a^2 - 1)^{3/2}}
\]
Problem 6: Evaluate the given integral by means of the residue theorem.

(a) \( \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} \) \quad (b) \( \int_{0}^{\infty} \frac{x}{x^4 + 1} \) \quad (c) \( \int_{0}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}, \quad a > 0, b > 0. \)

Solution: First in (a) \( z^4 + 1 = 0 \) has zeros at \( z_1 = e^{i\pi/4}, z_2 = e^{i3\pi/4} \) in the upper half plane. Thus using the same argument in the class

\[
\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = 2\pi i \sum_{j=1}^{2} \text{Res} \left( \frac{1}{1+z^4}; z_j \right)
\]

One can use the residue formula to calculate the residue. Another way to calculate the residue is using the Taylor series expansion

\[ 1 + z^4 = 4z_j^3(z - z_j) + \cdots, \quad z \to z_j \]

we obtain

\[
\frac{1}{1 + z^4} = \frac{1}{4z_j^3(z - z_j)} + \cdots = \frac{1}{4z_j} + \frac{1}{z - z_j} + \cdots, \quad z \to z_j.
\]

Hence,

\[
\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = 2\pi i \sum_{j=1}^{2} \left( -\frac{z_j}{4} \right) = -\frac{i\pi}{2} \left( e^{i\pi/4} + e^{i3\pi/4} \right) = \frac{\pi}{2} \sqrt{2}.
\]

In part (b) we first use substitution \( u = x^2 \) and obtain

\[
\int_{0}^{\infty} \frac{x \, dx}{x^4 + 1} = \frac{1}{2} \int_{0}^{\infty} \frac{du}{u^2 + 1} = \frac{1}{4} \int_{-\infty}^{\infty} \frac{du}{u^2 + 1}.
\]

The last integral can be computed with the help of \( \text{arctan}(x) \) but here we use residue theorem. Since \( z^2 + 1 = 0 \) has zero at \( z_1 = i \) in the upper half plane. Thus

\[
\int_{0}^{\infty} \frac{x \, dx}{x^4 + 1} = \frac{1}{4} \, 2\pi i \, \text{Res} \left( \frac{1}{1 + z^2}; i \right) = \frac{\pi}{4}.
\]
In part (c) we use evenness of the integrand

\[ \int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)} , \]

Since \((z^2 + a^2)(z^2 + b^2) = 0\) has no real roots and \(z_1 = ia, z_2 = ib\) are the only roots in the upper half plane. Thus

\[ \int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{1}{2} 2\pi i \sum_{j=1}^{2} \text{Res} \left( \frac{1}{(z^2 + a^2)(z^2 + b^2)}; z_j \right) \]

\[ = i\pi \left( \frac{1}{2ia} b^2 - a^2 + \frac{1}{2ib} a^2 - b^2 \right) = \frac{\pi}{2} \frac{1}{ab} \frac{1}{a+b} , \]

provided \(a \neq b\) (in this case \(z_1 \neq z_2\) and both poles are simple). If \(a = b\), then instead

\[ \int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = i\pi \left( -\frac{2}{(2ia)^3} \right) = \frac{\pi}{4a^3} \]

which matches the previous result in the limit \(a \to b\).

**Problem 7:** Evaluate the given integral by means of the residue theorem.

(a) \( \int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 1)^2} \, dx \)  
(b) \( \int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} \, dx, \ a > 0, b > 0, a \neq b. \)

**Solution:** In part (a) we have

\[ \int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 1)^2} \, dx = \Re \left\{ \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 1)^2} \, dx \right\} \]

and \((z^2 + 1)^2 = 0\) has zeros at \(z_1 = i\) in the upper half plane. Thus

\[ \int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 1)^2} \, dx = \Re \left\{ 2\pi i \text{Res} \left( \frac{e^{ix}}{(z^2 + 1)^2}; z_1 \right) \right\} \]
and since

\[
\text{Res} \left( \frac{e^{iz}}{(z^2 + 1)^2}, z_1 \right) = \frac{1}{2!} \left. \frac{d}{dz} \left[ \frac{e^{iz}}{(z^2 + 1)^2} (z - z_1)^2 \right] \right|_{z=z_1} = -\frac{i}{2e},
\]

we find

\[
\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 1)^2} \, dx = \Re \left\{ 2\pi i \left( -\frac{i}{2e} \right) \right\} = \frac{\pi}{e}.
\]

Start similarly in (b),

\[
\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} \, dx = \Re \left\{ \int_{-\infty}^{\infty} \frac{e^{iz}}{(x^2 + a^2)(x^2 + b^2)} \, dx \right\},
\]

since \((z^2 + a^2)(z^2 + b^2) = 0\) has no real roots and \(z_1 = ia, z_2 = ib\) are the only ones in the upper half plane. This leads us to

\[
\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} \, dx = \Re \left\{ 2\pi i \sum_{j=1}^{2} \text{Res} \left( \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)} ; z_j \right) \right\}
\]

\[
= \Re \left\{ 2\pi i \left( \frac{e^{-a}}{2ia(b^2 - a^2)} + \frac{e^{-b}}{2ib(a^2 - b^2)} \right) \right\}
\]

\[
= \frac{\pi}{b^2 - a^2} \left( \frac{e^{-a}}{a} - \frac{e^{-b}}{b} \right).
\]