Due date: 03/01. Please hand in your work at the beginning of the class.

**Exercise 1:** Find the Fourier cosine series of \( f(\theta) = \sin \theta, \ 0 < \theta < \pi \). Determine whether the Fourier cosine series converges to \( f \) or not.

**Exercise 2:** Apply the Abel’s or Dirichlet’s Test to show that the alternating harmonic series

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots
\]

is convergent.

**Exercise 3:** Consider the heat equation that models the heat flow with insulated endpoints

\[
\begin{align*}
    u_t &= ku_{xx}, \quad 0 < x < \ell, \ t > 0 \\
    u_x(0, t) &= u_x(\ell, t) = 0, \ t > 0 \\
    u(x, 0) &= f(x), \quad 0 < x < \ell.
\end{align*}
\]

Apply separation of variables to find a Fourier series solution.

**Exercise 4:** Solve the following wave equation using separation of variables

\[
\begin{align*}
    c^2 y_{xx} &= y_{tt}, \quad 0 < x < \ell, \ t > 0 \\
    y(0, t) &= 0, \quad y(\ell, t) = 0, \ t > 0 \\
    y(x, 0) &= 0, \quad y_t(x, 0) = 3 \sin\left(\frac{\pi x}{\ell}\right) - 5 \sin\left(\frac{4\pi x}{\ell}\right), \quad 0 < x < \ell.
\end{align*}
\]

**Exercise 5:** Consider the heat equation

\[
k u_{xx} = u_t + F(x, t), \quad 0 < x < L, \ t > 0
\]

with boundary conditions

\[
u(0, t) = p(t), \quad u(L, t) = q(t), \ t > 0
\]

and initial condition

\[
u(x, 0) = f(x), \quad 0 < x < L.
\]
(a) To establish the uniqueness of the solution, suppose that $u_1(x,t)$ and $u_2(x,t)$ are two solutions, and define $w(x,t) = u_1(x,t) - u_2(x,t)$. Show that $w$ satisfies the “homogenized” problem

$$kw_{xx} = w_t, \quad 0 < x < L, \ t > 0$$

with boundary conditions

$$w(0,t) = 0, \ w(L,t) = 0, \ t > 0$$

and initial condition

$$w(x,0) = 0, \ 0 < x < L.$$

(b) Define the energy $E(t) = \int_0^L |w(x,t)|^2 \, dx$. Show formally (assuming that you can differentiate $E$ and the function $w$) that $\frac{dE}{dt} \leq 0$. From above inequality, show that $w(x,t) = 0$ for all $x$ and $t$. Thus, it must be true that $u_1(x,t) = u_2(x,t)$ for any solutions $u_1$ and $u_2$, i.e. uniqueness follows.

**Exercise Bonus:** Verify the Fourier series solution to Exercise 3, if the initial temperature $f$ is continuous and piecewise smooth.