Please solve the problems and show all your work. Hand in your homework on Oct 11 class.

**Problem 1:** In each case solve the heat equation $\alpha^2 u_{xx} = u_t$ for $0 < x < L, t > 0$ with the boundary and initial conditions listed below. Use a half- or quarter-range cosine or sine expansion, as appropriate. Evaluate the expansion coefficients explicitly, rather than leaving them in integral form. Also, identify the steady-state solution $u_s(x)$.

(a) \[ \begin{cases} u(0, t) = 0, & u(10, t) = 100, & t > 0 \\ u(x, 0) = f(x) \equiv 0, & 0 < x < 10 \end{cases} \] (i.e. $L = 10$)

(b) \[ \begin{cases} u(0, t) = 10, & u_x(2, t) = -5, & t > 0 \\ u(x, 0) = f(x) \equiv 10, & 0 < x < 2 \end{cases} \] (i.e. $L = 2$)

**Problem 2:** Consider the diffusion equation $\alpha^2 u_{xx} = u_t + F(x, t), \ 0 < x < L, \ t > 0$ with Robin boundary conditions $u(0, t) = p(t), \ u(L, t) + \beta u_x(L, t) = q(t), \ t > 0$ for $\beta > 0$ and initial condition $u(x, 0) = f(x), \ 0 < x < L$.

(a) To establish the *uniqueness* of the solution, suppose that $u_1(x, t)$ and $u_2(x, t)$ are two solutions, and define $w(x, t) = u_1(x, t) - u_2(x, t)$. Show that $w$ satisfies the “homogenized” problem $\alpha^2 w_{xx} = w_t, \ 0 < x < L, \ t > 0$ with Robin boundary conditions $w(0, t) = 0, \ w(L, t) + \beta w_x(L, t) = 0, \ t > 0$ for $\beta > 0$ and initial condition $w(x, 0) = 0, \ 0 < x < L$. 

(b) Define the energy $E(t) = \int_0^L |w(x,t)|^2 dx$. Show that $\frac{dE}{dt} \leq 0$. From above inequality, show that $w(x,t) = 0$ for all $x$ and $t$. Thus, it must be true that $u_1(x,t) = u_2(x,t)$ for any solutions $u_1$ and $u_2$, i.e. uniqueness follows.

**Problem 3**: Section 18.3. Problem 10(c). (Steady-state solution) If the solution $u(x,t)$ tends to a steady-state solution $u_s(x)$ as $t \to \infty$, then we can determine $u_s(x)$ from $u(x,t)$ as

$$u_s = \lim_{t \to \infty} u(x,t).$$

However, if we are interested only in $u_s(x)$ then it is wasteful to first solve for $u(x,t)$. To solve for $u_s(x)$ directly, merely set $u_t = 0$ in the PDE, which step reduces the PDE to an ODE on $u_s(x)$. Solve that ODE subject to the boundary conditions (which, we assume here, do not vary with $t$). Use this method to find $u_s(x)$ in the following case:

$$\alpha^2 u_{xx} = u_t, \quad 0 < x < L, t > 0$$
$$u_x(0,t) = Q_1, \quad u_x(L,t) = Q_2,$$
$$u(x,0) = f(x).$$

You may follow the hint in the textbook.

**Problem 4**: Section 18.3. Problem 19. (Nonhomogeneous Neumann conditions) Consider the heat conduction problem

$$\alpha^2 u_{xx} = u_t, \quad 0 < x < L, t > 0$$
$$u_x(0,t) = -1, \quad u_x(L,t) = 0, \quad u(x,0) = 0.$$

You may find that the standard separation of variables procedure has difficulty coping with the boundary conditions since $u_x(o,t) \neq u_x(L,t)$. To proceed, we let

$$u(x,t) = \frac{(x-L)^2}{2L} + v(x,t).$$

Show that $v$ can be split by superposition into $v = v_1 + v_2$ where

$$v_1t - \alpha^2 v_{1xx} = 0, \quad 0 < x < L, t > 0,$$
$$v_1(0,t) = 0, \quad v_1(L,t) = 0, \quad v_1(x,0) = -\frac{(x-L)^2}{2L},$$
$$v_2t - \alpha^2 v_{2xx} = F(x,t) = \alpha^2/L, \quad 0 < x < L, t > 0,$$
$$v_2(0,t) = 0, \quad v_2(L,t) = 0, \quad v_2(x,0) = 0.$$

Find the solution for $v_2$ (you may use the eigenfunctions of an appropriate Sturm-Liouville problem).