Homework 5 solution

Exercise 1: Suppose \( \{ \phi_n \} \) is an orthonormal basis for \( L^2(a, b) \). Show that for any \( f, g \in L^2(a, b) \),

\[
\langle f, g \rangle = \sum \langle f, \phi_n \rangle \overline{\langle g, \phi_n \rangle}.
\]

Solution: Since \( \{ \phi_n \} \) is an orthonormal basis for \( L^2(a, b) \), then

\[
f(x) = \sum \langle f, \phi_n \rangle \phi_n(x), \quad g(x) = \sum \langle g, \phi_m \rangle \phi_m(x).
\]

Therefore by the orthogonality of \( \{ \phi_n \} \),

\[
\langle f, g \rangle = \sum \langle f, \phi_n \rangle \overline{\langle g, \phi_m \rangle} \delta_{mn} = \sum \langle f, \phi_n \rangle \overline{\langle g, \phi_n \rangle}.
\]

Exercise 2: Show that \( \{ e^{2\pi i(mx + ny)} \}_{m,n=-\infty}^\infty \) is an orthonormal set in \( L^2(D) \) where \( D = (0, 1) \times (0, 1) \) is a square.

Solution: For any integers \( m, n, p, q \), we can directly compute

\[
\int_0^1 \int_0^1 e^{2\pi i(mx + ny)} e^{2\pi i(px + qy)} \, dx \, dy = \int_0^1 e^{2\pi i(m+p)x} \, dx \int_0^1 e^{2\pi i(n+q)y} \, dy = \delta_{mp} \delta_{nq}.
\]

This means that \( e^{2\pi i(mx + ny)} \) is orthonormal to \( e^{2\pi i(px + qy)} \) if \( (m, n) \neq (p, q) \), and \( e^{2\pi i(mx + ny)} \) has unit norm. Hence \( \{ e^{2\pi i(mx + ny)} \}_{m,n=-\infty}^\infty \) is an orthonormal set in \( L^2(D) \).

Exercise 3: Identity \( r(x), p(x), w(x), \alpha, \beta, \gamma, \delta \), solve for the eigenvalues and eigenfunctions, and use the eigenfunctions to write down the generalized Fourier expansion of a function \( f \in L^2(0, L) \).

\[
y'' + \lambda y = 0, \quad 0 < x < L, y(0) = 0, y'(L) = 0.
\]
**Solution:** Here $r(x) = 1$, $p(x) = 0$, $w(x) = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$ and $\delta = 1$. We look for solutions in the form

$$y(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x), \quad \lambda \neq 0,$$

and

$$y(x) = C + Dx, \quad \lambda = 0.$$

When $\lambda = 0$, the boundary condition gives

$$C = 0, \quad D = 0.$$  

This gives a trivial solution. Then we consider $\lambda \neq 0$. The boundary condition gives

$$A = 0, \quad -A\sqrt{\lambda} \sin(\sqrt{\lambda}L) + B\sqrt{\lambda} \cos(\sqrt{\lambda}L) = 0,$$

from which we can get $A = 0$ and

$$\cos(\sqrt{\lambda}L) = 0.$$

Then we have $\sqrt{\lambda} = \frac{n\pi}{2L}$, $n = 1, 3, 5, \cdots$. Now the eigenvalue is $\lambda_n = \left(\frac{n\pi}{2L}\right)^2$ and the corresponding eigenfunction is $\phi_n(x) = \sin(\frac{n\pi x}{2L})$. Now from generalized Fourier expansion

$$f(x) = \sum_{n=1,3,\cdots} a_n \sin\left(\frac{n\pi x}{2L}\right),$$

where

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_0^L f(x) \sin\left(\frac{n\pi x}{2L}\right)dx}{\int_0^L \sin^2\left(\frac{n\pi x}{2L}\right)dx} = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{2L}\right)dx.$$

**Exercise 4:** (Obtaining Sturm-Liouville form) The equation

$$A(x)y'' + B(x)y' + C(x)y + \lambda D(x)y = 0, \quad a < x < b$$

is in the standard Sturm-Liouville form only if $B(x) = A'(x)$. If $A(x) \neq 0$ on $[a,b]$ and $(B - A')/A$ is continuous on $[a,b]$, we may recast (1) in the form

$$[r(x)y']' + p(x)y + \lambda w(x)y = 0, \quad a < x < b$$

by multiplying equation (1) by $\sigma(x) = e^{\int[(B-A')/A]dx}$. Use the above method to recast the following problem in the standard Sturm-Liouville form

$$x^2 y'' + xy' + \lambda y = 0, \quad 1 < x < a,$$

$$y(1) = 0, \quad y(a) = 0.$$
**Solution:** We multiply $\sigma$ to the equation and get

$$A\sigma y'' + B\sigma y' + Cy + \lambda Dy = 0.$$ 

The regular Sturm-Liouville problem is of the form

$$(ry')' + py + \lambda wy = 0,$$

that is

$$ry'' + r'y' + py + \lambda wy = 0.$$ 

We now require that

$$A\sigma = r, \quad B\sigma = r',$$

and this gives

$$(A\sigma)' = B\sigma.$$ 

The above equations is further simplified to

$$A\sigma' = (B - A')\sigma.$$ 

This is a first-order linear ODE, if $A(x) \neq 0$ and $\frac{B - A'}{A}$ is continuous on $[a, b]$, we know that one solution is

$$\sigma(x) = e^{\int \frac{B - A'}{A} dx}.$$ 

Now for the problem

$$x^2y'' + xy' + \lambda y = 0, \quad 1 < x < a,$$

$$y(1) = 0, \quad y(a) = 0,$$

we have

$$\sigma(x) = e^{\int \frac{B - A'}{A} dx} = e^{\int \frac{x - (x^2/2)'}{x^2} dx} = e^{\int \frac{-1}{x} dx} = e^{-\ln(x)} = \frac{1}{x}.$$ 

**Exercise 5:** Consider the eigenvalue problem

(2) $$x^2y'' + xy' + \lambda y = 0, \quad 1 < x < a,$$

(3) $$y(1) = 0, \quad y(a) = 0.$$
Show that the eigenvalues and eigenfunctions are

\[ \lambda_n = \left( \frac{n\pi}{\ln a} \right)^2, \quad n = 1, 2, \ldots \]

\[ \phi_n(x) = \sin \left( \frac{n\pi \ln x}{\ln a} \right), \quad n = 1, 2, \ldots. \]

Moreover, write down the generalized Fourier expansion of a given function \( f \in L^2(0, a) \).

**Solution:** To find eigenvalues and eigenvectors, we need to solve the equation and find which \( \lambda \)'s admit non-zero solutions with given boundary conditions. We start to work with the original equation

\[(4) \quad x^2 y'' + xy' + \lambda y = 0.\]

By substituting \( y(x) = z(\ln x) \), we can write this equation as

\[ z''(\ln x) + \lambda z(\ln x) = 0. \]

Thus we need to solve

\[ z'' + \lambda z = 0 \]

to find the solutions to the original equation. The solutions are

\[ z(t) = \begin{cases} 
A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t), & \lambda > 0 \\
C + Dt, & \lambda = 0
\end{cases} \]

Thus the solutions of (4) are

\[ y(x) = \begin{cases} 
A \cos(\sqrt{\lambda} \ln x) + B \sin(\sqrt{\lambda} \ln x), & \lambda > 0 \\
C + D \ln x, & \lambda = 0
\end{cases} \]

When \( \lambda = 0 \), the boundary conditions give \( C = D = 0 \). Then we consider \( \lambda \neq 0 \). The boundary conditions give

\[ 0 = y(1) = A \]
\[ 0 = y(a) = A \cos(\sqrt{\lambda} \ln a) + B \sin(\sqrt{\lambda} \ln a). \]

To have a non-trivial solution we require that

\[ A = 0, \quad \sqrt{\lambda} \ln a = n\pi, \quad n = \pm 1, \pm 2, \ldots. \]
Then the eigenvalues are

$$\lambda = \left( \frac{n\pi}{\ln a} \right)^2, \quad n = 1, 2, \ldots$$

and the corresponding eigenfunctions are

$$\phi_n(x) = \sin \left( \frac{n\pi}{\ln a} \ln x \right) = \sin \left( n\pi \frac{\ln x}{\ln a} \right).$$

The generalized Fourier expansion is

$$f(x) = \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n(x),$$

where

$$\langle f, \phi_n \rangle = \int_1^a f(x) \phi_n(x) w(x) dx = \int_1^a f(x) \sin \left( n\pi \frac{\ln x}{\ln a} \right) \frac{1}{x} dx$$

and

$$\langle \phi_n, \phi_n \rangle = \int_1^a \phi_n(x) \phi_n(x) w(x) dx = \int_1^a \sin^2 \left( n\pi \frac{\ln x}{\ln a} \right) \frac{1}{x} dx.$$