**Problem 2:** Solve the Laplace equation using separation of variables

\[
\begin{align*}
    u_{xx} + u_{yy} &= 0, \quad 0 < x < 3, 0 < y < 2, \\
    u(0, y) &= u(x, 2) = u_x(3, y) = 0, u(x, 0) = 50H(x - 2),
\end{align*}
\]

where \( H(x) \) is the Heaviside function

\[
H(x) = \begin{cases} 
    0, & x \leq 0 \\
    1, & x > 0.
\end{cases}
\]

**Solution:** Seek solutions in the form \( u(x, y) = X(x)Y(y) \). This gives

\[
X''Y + XY'' = 0.
\]

Since \( u(0, y) = u_x(3, y) = 0 \), we expect that The first equation gives

\[
\frac{X''}{X} = -\frac{Y''}{Y} = -\kappa^2 \text{(constant)},
\]

and this gives

\[
\begin{align*}
    X'' + \kappa^2 X &= 0, \\
    Y'' - \kappa^2 Y &= 0.
\end{align*}
\]

Now \( X \) and \( Y \) are solved as

\[
X = \begin{cases} 
    A + Bx & \kappa = 0, \\
    C \cos \kappa x + D \sin \kappa x & \kappa \neq 0.
\end{cases}
\]

\[
Y = \begin{cases} 
    E + Fy & \kappa = 0, \\
    G \cosh \kappa y + H \sinh \kappa y & \kappa \neq 0.
\end{cases}
\]
Now we have that
\[ u(x, y) = (A + Bx)(E + Fy) + (C \cos \kappa x + D \sin \kappa x)(G \cosh \kappa y + H \sinh \kappa y). \]

Boundary condition \( u(0, y) = u_x(3, y) = 0 \) yields
\[
0 = (A)(E + Fy) + C(G \cosh \kappa y + H \sinh \kappa y),
\]
\[
0 = B(E + Fy) + (-C \kappa \sin 3 + D \kappa \cos 3)(G \cosh \kappa y + H \sinh \kappa y).
\]

To have a robust solution we have that \( A = 0, \ C = 0, \ B = 0, \ \cos 3\kappa = 0, \) and consequently \( \kappa = \frac{(2n-1)\pi}{6}, n = 1, 2, \cdots. \)

Now from superposition principle
\[
u(x, y) = \sum_{n=1}^{\infty} \sin \left( \frac{(2n-1)\pi x}{6} \right) \left( I_n \cosh \left( \frac{(2n-1)\pi y}{6} \right) + J_n \sinh \left( \frac{(2n-1)\pi y}{6} \right) \right).
\]

Boundary condition \( u(x, 2) = 0 \) yields
\[
0 = \sum_{n=1}^{\infty} \sin \left( \frac{(2n-1)\pi x}{6} \right) \left( I_n \cosh \left( \frac{(2n-1)\pi y}{6} \right) + J_n \sinh \left( \frac{(2n-1)\pi y}{6} \right) \right).
\]

Boundary condition \( u(x, 0) = 0 \) yields
\[
50H(x - 2) = \sum_{n=1}^{\infty} \sin \left( \frac{(2n-1)\pi x}{6} \right) I_n.
\]

From QRS expansion
\[
0 = I_n \cosh \left( \frac{(2n-1)\pi y}{6} \right) + J_n \sinh \left( \frac{(2n-1)\pi y}{6} \right),
\]
\[
I_n = \frac{2}{3} \int_{0}^{3} 50H(x - 2) \sin \left( \frac{(2n-1)\pi x}{6} \right) dx = \frac{100}{3} \frac{6}{\pi(2n-1)} \cos \frac{\pi(2n-1)}{3}.
\]
Solving the above linear system gives

\[ I_n = \frac{100}{3} \frac{6}{\pi(2n-1)} \cos \frac{\pi(2n-1)}{3}, \]

\[ J_n = -\frac{\cosh \frac{(2n-1)\pi}{3}}{\sinh \frac{(2n-1)\pi}{3}} I_n. \]

With the above formula for \( I_n \) and \( J_n \), the solution is

\[ u(x, y) = \sum_{n=1}^{\infty} \sin \left( \frac{(2n-1)\pi x}{6} \right) \left( I_n \cosh \frac{(2n-1)\pi y}{6} + J_n \sinh \frac{(2n-1)\pi y}{6} \right). \]

**Problem 3:** Consider a thin flat plate of radius \( b \), that is thermally insulated on its two flat faces. With a hacksaw we make a radial cut along \( \theta = 0 \), say, from \( r = b \) to \( r = 0 \). The small gap, due to the cut, may be approximated bas a thermal insulator, so that \( u_\theta = 0 \) on the edges \( \theta = 0 \) and \( \theta = 2\pi \). If the circumference of the plate is held at the temperature \( 50(1 + \sin \theta) \) for a long time, the steady-state temperature field \( u(r, \theta) \) is governed by the boundary-value problem

\[ \nabla^2 u = 0, \quad 0 < r < b, 0 < \theta < 2\pi, \]

\[ u_\theta(r, 0) = u_\theta(r, 2\pi) = 0, \]

\[ u(b, \theta) = 50(1 + \sin \theta), \]

\[ u \text{ bounded.} \]

Solve for \( u(r, \theta) \).

**Solution:** The Laplace equation in Polar coordinates reads

\[ \nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}. \]

Suppose that \( u(r, \theta) = R(r)\Theta(\theta) \) solves the equation. Separation of variables give equations

\[ r^2 R''(r) + r R'(r) - k^2 R(r) = 0 \]

\[ \Theta''(\theta) + k^2 \Theta(\theta) = 0. \]
Solving for Θ gives

\[ Θ(θ) = \begin{cases} 
Iθ + J, & k = 0 \\
K \cos(kθ) + L \sin(kθ), & k \neq 0.
\end{cases} \]

Thus the boundary conditions for \( u \) give that

\[ 0 = Θ'(0) = Θ'(2\pi). \]

Thus we get that \( Θ(θ) = J \) is a constant when \( k = 0 \). When \( k \neq 0 \) we have that

\[ 0 = Lk \rightarrow L = 0, \]

\[ 0 = -Kk \sin(k2\pi) \rightarrow k = \frac{n}{2}, n = 1, 2, \cdots. \]

Now we have that

\[ Θ(θ) = K_n \cos\left(\frac{n}{2}θ\right) \]

where \( n \) is a positive integer.

Since \( R(r) \) is bounded, we get (from the example that was covered in the class) that

\[ R(r) = B_0(k = 0) \] or \( R(r) = B_n r^{n/2}(k \neq 0) \).

Hence from superposition of principle

\[ u(r, θ) = D + \sum_{n=1}^{∞} A_n r^{n/2} \cos\left(\frac{n}{2}θ\right). \]

Since \( u(b, θ) = 50(1 + \sin θ) = f(θ) \), we now use the HRC extension of \( f(θ) \) (\( L = 2\pi \)):

\[ f(θ) = a_0 + \sum_{n=1}^{∞} a_n \cos\left(\frac{nπ}{2\pi}θ\right), \]

where

\[ a_0 = \frac{1}{2\pi} \int_{0}^{2\pi} 50(1 + \sin θ)dθ = 50. \]
and, by integration by parts,

\[ a_n = \frac{2}{2\pi} \int_0^{2\pi} 50(1 + \sin \theta) \cos \left( \frac{n}{2} \theta \right) d\theta = \frac{50}{\pi} \frac{1 - (-1)^n}{1 - (n/2)^2} = \begin{cases} 0, & n \text{ even} \\ \frac{100}{\pi} \frac{1}{1 - (n/2)^2}, & n \text{ odd} \end{cases} \]

Thus we have that \( D = a_0 \) and \( A_n b^{n/2} = a_n \) for \( n \) odd, and \( A_n = 0 \) for \( n \) even.

Finally we obtain that

\[ u(r, \theta) = 50 + \sum_{n>0 \text{ odd}}^{\infty} \frac{100}{b^{n/2}\pi} \frac{1}{1 - (n/2)^2} r^{n/2} \cos \left( \frac{n}{2} \theta \right). \]

**Problem * (extra 1/5 of this assignment set):** Consider the Poisson equation

\[ \nabla^2 u = f(x, y, z) \]

in some three-dimensional domain \( D \) with surface \( S \). Integrating the Poisson equation over \( D \), show that the following solvability condition holds

\[ \int_S \frac{\partial u}{\partial n} dA = \int_D f dV, \]

where \( n \) is the unit outward normal on \( \partial D \). You may use Green’s identity.

**Solution:** From the Poisson equation and integration over \( D \)

\[ \int_D f dV = \int_D \nabla^2 u dV. \]

From Green’s identity (or you may also use divergence Theorem)

\[ \int_D \nabla^2 u dV = \int_S \frac{\partial u}{\partial n} 1 dA - \int_D \nabla u \cdot \nabla 1 dV = \int_S \frac{\partial u}{\partial n} dA. \]