Problem 1: (a) Show that $\frac{1}{z} = \frac{1}{\bar{z}}$. (b) Evaluate $\text{Im} \frac{a + ib}{c + id}$. (c) Evaluate $|(1 - 2i)^2 + (1 + i)^2|$. (d) Show whether or not $|e^z| = e^{|z|}$.

Solutions:

(a):

$$\frac{\bar{z}(\frac{1}{z})}{(\frac{1}{z})} = \frac{\frac{1}{z}}{\frac{1}{z}} = \bar{1} = 1 \Rightarrow \frac{1}{z} = \frac{1}{\bar{z}}.$$ 

(b):

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}$$

$$\Rightarrow \text{Im} \frac{a + ib}{c + id} = \frac{bc - ad}{c^2 + d^2}.$$ 

(c):

$$|(1 - 2i)^2 + (1 + i)^2| = 1 - 4i - 4 + 1 + 2i - 1 = -3 - 2i$$

$$\Rightarrow |(1 - 2i)^2 + (1 + i)^2| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}.$$ 

(d): Let $z = iy$ where $y$ is real, then

$$|e^z| = |\cos y + i\sin y| = 1,$$

$$e^{|z|} = e^{|y|},$$

then in general $|e^z| = e^{|z|}$ is not true.

Problem 2: Recall that we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$
(a) Show that \( \sin(-z) = -\sin z \).  
(b) Show that \( \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \).  
(c) Show that \( \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y \), where \( x \) and \( y \) are real.

**Solutions:**

(a):

\[
\sin(-z) = \frac{e^{i(-z)} - e^{-i(-z)}}{2i} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z.
\]

(b):

\[
\cos(z_1) \cos(z_2) - \sin(z_1) \sin(z_2) = \frac{1}{4} \left( e^{iz_1} + e^{-iz_1} \right) \left( e^{iz_2} + e^{-iz_2} \right) \\
- \frac{1}{4i^2} \left( e^{iz_1} - e^{-iz_1} \right) \left( e^{iz_2} - e^{-iz_2} \right) \\
= \frac{1}{4} \left( e^{i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{i(z_2-z_1)} + e^{-i(z_1+z_2)} \right) \\
+ \frac{1}{4} \left( e^{i(z_1+z_2)} - e^{i(z_1-z_2)} - e^{i(z_2-z_1)} - e^{-i(z_1+z_2)} \right) \\
= \frac{1}{2} \left( e^{i(z_1+z_2)} + e^{-i(z_1+z_2)} \right) = \cos(z_1 + z_2).
\]

(c) From subproblem (b), it is sufficient to show that \( \cos(iy) = \cosh(y) \). Indeed

\[
\cos(iy) = \frac{e^{iy} + e^{-iy}}{2} = \frac{e^y + e^{-y}}{2} = \cosh(y).
\]

**Problem 3:** (a) Recall that the argument of \( z = x + iy \) can be determined by \( \arg(z) = \arctan \frac{y}{x} \) with the help of showing the complex number in the complex plane. Find all values of \( z^{1/2} \) and \( z^{1/5} \) for \( z = 1 \) and \( z = 3 - 2i \), express those values in polar form. (b) Obtain all values of \( \log z \) for \( z = 1 \) and \( z = 2 - i \).

**Solution:** In the solution, \( n \) is an integer.
(a) $1 = 1e^{0}$, then $1^{1/2} = 1e^{i(0+n\pi)}$ where $n$ is integer, this gives that $1^{1/2} = \pm 1$ and similarly $1^{1/5} = 1e^{i(0+2n\pi/5)} = 1, e^{i\pi/5}, e^{i\pi/5}, e^{i\pi/5}, e^{i\pi/5}$, the roots are evenly spaced on the unit circle.

Consider $z = 3 - 2i$, we have $r = |3 - 2i| = \sqrt{9 + 4} = \sqrt{13}$, with the help of showing the complex number in the complex plane we find that $\theta = \arg(3 - 2i) = -\arctan(2/3) \in (0, \pi/2)$.

Then $(3 - 2i)^{1/2} = r^{1/2}e^{i(\theta+2n\pi)/2}$ where $n$ is integer, this gives that $(3 - 2i)^{1/2} = 13^{1/4}e^{-i\arctan(2/3)/2}, 13^{1/4}e^{i(-\arctan(2/3)/2+\pi)}$ and similarly $(3 - 2i)^{1/5} = r^{1/5}e^{i(\theta+2n\pi)/5}$, the roots are evenly spaced on the unit circle.

(b)

\[
\log 1 = \log 1e^{0} = \ln 1 + in\pi = in\pi,
\]

\[
\log(2 - i) = \log \sqrt{5}e^{-i\arctan(1/2)} = \ln 5/2 + i(-\arctan(1/2) + n2\pi),
\]

where $n$'s are integers and $\arctan(1/2) \in (0, \pi/2)$.

**Problem 4:** One can define

\[c^z = e^{z(ln|c|+ic)}.\]

Use this to evaluate $(1 + \sqrt{3}i)^{(2-5i)}$.

**Solutions:**

\[c = 1 + \sqrt{3}i \quad \Rightarrow |c| = \sqrt{1+3} = 2, \quad \arg c = \arctan \sqrt{3} = \pi/3 \quad \Rightarrow c = 2e^{i\pi/3}.\]

This gives

\[
c^z = e^{z(ln|c|+ic)} = e^{(2-5i)(ln 2+i\pi/3)} = 4e^{5\pi/3}e^{i2\pi/3}e^{-i\pi/3} = 2e^{5\pi/3}(-1 + i\sqrt{3})e^{-i\pi/3} = 2e^{5\pi/3}(-\cos(5 \ln 2) + i\sin(5 \ln 2)) = 2e^{5\pi/3}(-\cos(5 \ln 2) + \sqrt{3}\sin(5 \ln 2) + i(\sqrt{3}\cos(5 \ln 2) + \sin(5 \ln 2))).\]

**Problem 5:** Given $f(z)$, determine $f'(z)$, where it exists, and state where $f$ is differentiable and where it is not.

(a) \[\frac{1}{z^3 + 1}\]  
(b) $z \sin z$
**Solution:** (a) The function \( f(z) = \frac{1}{z^3 + 1} \) is defined for \( z \in \mathbb{C} \setminus \{e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}\} \) and if we fix \( z_0 \in \mathbb{C} \setminus \{e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}\} \) then

\[
\frac{f(z) - f(z_0)}{z - z_0} = -\frac{z^3 - z_0^3}{(z - z_0)(z^3 + 1)(z_0^3 + 1)} = -\frac{z^2 + zz_0 + z_0^2}{(z^3 + 1)(z_0^3 + 1)} \xrightarrow{z \to z_0} \frac{3z_0^2}{(z_0^3 + 1)^2}.
\]

Hence \( f'(z_0) \) exists and

\[
f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = -\frac{3z_0^2}{(z_0^3 + 1)^2}.
\]

In summary, \( f \) is differentiable for all \( z_0 \in \mathbb{C} \setminus \{e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}\} \) with derivative given in equation (1) above.

(b) \( g(z) = z \sin z \) is defined for all \( z \in \mathbb{C} \). Write \( z = x + iy \) and we can obtain that \( f(z) = u(x, y) + iv(x, y) \) where

\[
u(x, y) = x \cos x \sinh y + y \sin x \cosh y.
\]

Now

\[
u_x = \sin x \cosh y + x \cos x \cosh y + y \sin x \sinh y \\
v_y = x \sin x \sinh y - \cos x \sinh y - y \cos x \cosh y
\]

Thus \( u_x = v_y \) and \( u_y = -v_x \) for all \( z \in \mathbb{C} \), i.e. the Cauchy Riemann equations are satisfied. Then \( f(z) \) is differentiable in the complex plane and

\[
f'(z) = u_x + iv_x = \sin x \cosh y + x \cos x \cosh y + y \sin x \sinh y \\
+ i(\cos x \sinh y - x \sin x \sinh y + y \cos x \cosh y) = \sin z + z \cosh z.
\]