Lecture 24: Residue Theorem and Applications

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A function $f$ has an isolated singularity at $z = a$ if there is $R > 0$ such that $f$ is defined and analytic in $0 < |z - a| < R$ but not in $0 \leq |z - a| < R$; otherwise it is said that $f$ has a non-isolated singular point at $z = a$. 
Let \( z = a \) be an isolated singularity of \( f \) and

\[
f(z) = \sum_{n=\infty}^{-\infty} a_n(z-a)^n, \quad 0 < |z-a| < R.
\]

Then

- \( z = a \) is a removable singularity iff \( a_n = 0 \) for \( n \leq -1 \).
- \( z = a \) is a pole of order \( m \) iff \( a_{-m} \neq 0 \) and \( a_n = 0 \) for \( n \leq -(m+1) \).
- \( z = a \) is an essential singularity iff \( a_n \neq 0 \) for infinitely many negative integers \( n \).
Residue Theorem

Consider the contour integral

\[ I = \int_C f(z)\,dz, \]

where \( C \) is a closed, counter-clockwise, and \( f(z) \) is analytic inside and on \( C \) except at finitely many isolated points \( z_1, z_2, \ldots, z_k \) within \( C \). If \( c^{(j)} \) denotes the residue of \( f \) at \( z_j \), then

\[ \int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} c^{(j)} = 2\pi i \sum_{j=1}^{k} \text{Res}(f; z_j). \]

What are residues? How to compute them?
Suppose $f$ has the following series expansion at $z_j$

$$f(z) = \sum_{n=\infty}^{\infty} c_n^{(j)} (z - z_j)^n, \quad 0 < |z - z_j| < \rho_j$$

for some $\rho_j$. Then the residue at $z_j$ is given by

$$\text{Res}(f; z_j) = c_{-1}^{(j)}.$$
Suppose \( f(z) \) has the following series expansion at \( z = a \),

\[
f(z) = c_{-N} \frac{1}{(z-a)^N} + c_{-N+1} \frac{1}{(z-a)^{N-1}} + \cdots
\]

\[
= \sum_{-N}^{\infty} c_n^{(j)} (z-a)^n, \quad 0 < |z - z_j| < \rho_j,
\]

then the residue is given by

\[
c_{-1} = \frac{1}{(N-1)!} \lim_{z \to a} \left( \frac{d^{N-1}}{dz^{N-1}} [(z-a)^N f(z)] \right).
\]
Example

\[ l = \int_{|z|=1} z^4 \sin \frac{1}{z} \, dz. \]
Example
Evaluate the residues of $f(z)$ at $z = -4$ and $z = 1$ where

$$f(z) = \frac{1}{(z + 4)(z - 1)^3}.$$
Example

Evaluate the integral

\[ I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}. \]
Example
Evaluate the integral

\[ I = \int_0^\infty \frac{\cos(ax)}{x^2 + 1} \, dx, \quad a > 0. \]
Example

Evaluate the integral

\[ I = \int_0^{2\pi} \frac{d\theta}{2 - \sin \theta}. \]