Lecture 1: Review of ODEs

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Find the general solution to

\[ a_0 \frac{dy^n}{dx^n} + a_1 \frac{dy^{n-1}}{dx^{n-1}} + \cdots + a_n y = f. \]

1. First-order (non-)homogeneous ODE
2. Second-order homogeneous ODE
3. High-order homogeneous ODE, linear independence
4. Second-order non-homogeneous ODE
Question: Find the general solution to 

\[ y' + p(x)y = 0. \]

Solution: The general solution is 

\[ y(x) = Ae^{-\int_x^p(\xi)d\xi}, \]

where \( A \) is an arbitrary constant.

Why? Rewrite the equation (separation of variables) as

\[ \frac{y'}{y} = -p(x), \]

\[ (\ln y)' = -p(x). \]

\[ y = Ae^{-\int_x^p(\xi)d\xi} \]
First-order non-homogeneous ODE

**Question**: Find the general solution to

\[ y' + p(x)y = q(x). \]

**Solution**: The general solution is

\[ y = e^{-\int^x p(\xi)d\xi} \left[ \int^x \int^{\sigma} e^{p(\xi)d\xi} q(\sigma)d\sigma + c \right], \]

where \( c \) is an arbitrary constant.

There are two approaches:

1. Variation of parameters \( y = A(x)e^{-\int^x p(\xi)d\xi}. \)
2. Rewrite the equation as

\[ \sigma(x)y' + \sigma(x)p(x)y = \sigma(x)q(x), \]

where \( \sigma(x) \) is such that

\[ (\sigma y)' = \sigma(x)y' + \sigma(x)p(x)y. \]
Suppose $y_1$ and $y_2$ satisfies equations

$$y_1' + p(x)y_1 = q(x),$$
$$y_2' + p(x)y_2 = q(x).$$

Taking the difference, $y = y_1 - y_2$ satisfies

$$y' + p(x)y = 0,$$

and zero boundary conditions. This gives that $y = 0$ and hence $y_1 = y_2$. 

Uniqueness of solution
Second order ODE

\[ y'' + a_1 y' + a_2 y = 0. \]

If we look for solutions as \( y = e^{\lambda x} \), this gives that

\[ \lambda^2 + a_1 \lambda + a_2 = 0. \]

There are two roots \( \lambda_1 \) and \( \lambda_2 \).

**Example**

Find the general solution to

\[ y'' - 2y' - 8y = 0. \]
Let us look at another example.

**Example**
Find the general solution to

$$y'' - 6y' + 9y = 0.$$  

Hint: Can $y = c(x)e^{\lambda x}$ ($c(x)$ is to be determined) be a particular solution?
$y'' + a_1 y' + a_2 y = 0$.

If we look for solutions as $y = e^{\lambda x}$, this gives that

$$\lambda^2 + a_1 \lambda + a_2 = 0.$$ 

There are two roots $\lambda_1$ and $\lambda_2$.

1. If $\lambda_1 \neq \lambda_2$, then $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

2. If $\lambda_1 = \lambda_2 = \lambda$, then $y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$. 
Let $\lambda_1, \cdots, \lambda_n$ be any numbers. Then $\{e^{\lambda_k x}\}_{k=1}^n$ is linear independent if and only if the $\lambda$’s are distinct.

If $\lambda$ is a root of order $k$, then $e^{\lambda x}, xe^{\lambda x} \cdots, x^{k-1}e^{\lambda x}$ are $k$ linear independent solutions.
Homogeneous high order ODE

\[ a_0 \frac{dy^n}{dx^n} + a_1 \frac{dy^{n-1}}{dx^{n-1}} + \cdots + a_n y = 0. \]

If we look for solutions as \( y = e^{\lambda x} \), this gives that

\[ a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0. \]

If we have a root \( \lambda \) of order \( k \), then \( e^{\lambda x}, xe^{\lambda x}, \cdots, x^{k-1}e^{\lambda x} \) are \( k \) linear independent solutions.

If we have (the remaining roots) \( \lambda_1, \cdots, \lambda_n \) that are distinct, then \( \{ e^{\lambda_k x} \}_{k=1}^n \) is another set of linear independent solutions.
Cauchy-Euler equation

\[ a_0 x^n \frac{dy^n}{dx^n} + a_1 x^{n-1} \frac{dy^{n-1}}{dx^{n-1}} + \cdots + a_n y = 0. \]

If we do a change of variable \( \xi = \ln x \), then

\[ b_0 \frac{dy^n}{d\xi^n} + b_1 \frac{dy^{n-1}}{d\xi^{n-1}} + \cdots + b_n y = 0. \]
Non-homogeneous high order ODE

General solution of

\[ Ly = f, \]

where

\[ L = a_0 \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n. \]

Let \( y_h \) be the solution to the homogeneous equation \( Ly_h = 0 \), and \( y_p \) be a particular solution to \( Ly_p = f \), then the general solution is

\[ y = y_h + y_p. \]
General solution of

\[ Ly = f_1 + \cdots + f_k, \]

is

\[ y = y_h + y_{p1} + \cdots + y_{pk}, \]

where \( y_h \) is the solution to the homogeneous equation \( Ly_h = 0 \), and \( y_{p\ell} \) is a particular solution to

\[ Ly_{p\ell} = f_\ell. \]
Non-homogeneous second order ODE

\[ y'' + a_1 y' + a_2 y = f(x). \]

Homogeneous solution is

\[ y_h(x) = c_1 y_1(x) + c_2 y_2(x), \]

where \( c_1 \) and \( c_2 \) are constants.

If we look for particular solution as

\[ y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x), \]

where \( c_1(x) \) and \( c_2(x) \) are functions.
Non-homogeneous second order ODE (continued)

\[ y'_p(x) = c'_1(x)y_1(x) + c'_2(x)y_2(x) + c_1(x)y'_1(x) + c_2(x)y'_2(x), \]

and require that

\[ c'_1(x)y_1(x) + c'_2(x)y_2(x) = 0. \]

Continue to calculate \( y''_p(x) \),

\[ y''_p(x) = c'_1(x)y'_1(x) + c'_2(x)y'_2(x) + c_1(x)y''_1(x) + c_2(x)y''_2(x). \]

Then

\[ y''_p + a_1y'_p + a_2y_p = c'_1(x)y'_1(x) + c'_2(x)y'_2(x), \]

from which we require that

\[ c'_1(x)y'_1(x) + c'_2(x)y'_2(x) = f. \]
Non-homogeneous second order ODE (continued)

Solve $c_1$ and $c_2$ such that

\[ c'_1(x)y_1(x) + c'_2(x)y_2(x) = 0, \]
\[ c'_1(x)y'_1(x) + c'_2(x)y'_2(x) = f. \]

This gives

\[
\begin{align*}
  c'_1(x) &= \frac{-fy_2}{y_1y'_2 - y'_1y_2}, \\
  c'_2(x) &= \frac{fy_1}{y_1y'_2 - y'_1y_2}.
\end{align*}
\]

Finally

\[
y_p(x) = \left[ \int^x c'_1(\xi)d\xi \right]y_1(x) + \left[ \int^x c'_2(\xi)d\xi \right]y_2(x).\]
Example

\[ y'' - 4y = 8e^{2x} \]