Lecture 11: D’Alembert’s Solution and Laplace Equation

Shixu Meng, Department of Mathematics, University of Michigan
Consider the 1-d wave equation

\[ c^2 y_{xx} = y_{tt}, \quad -\infty < x < \infty, \quad t > 0, \]
\[ y(x, 0) = f(x), \quad y_t(x, 0) = g(x), \quad -\infty < x < \infty. \]

Let \( y(x, t) = F(x - ct) + G(x + ct) \), then the initial conditions give

\[ F(x) + G(x) = f(x), \quad -cF'(x) + cG'(x) = g(x). \]

To solve for \( F(x) \) and \( G(x) \), first we note that

\[ \int_0^x g(\xi)d\xi = -cF(x) + cG(x) - ( -cF(0) + cG(0) ). \]

Then

\[ F(x) = \frac{f(x)}{2} - \frac{1}{2c} \int_0^x g(\xi)d\xi + \frac{F(0) - G(0)}{2}, \]
\[ G(x) = \frac{f(x)}{2} + \frac{1}{2c} \int_0^x g(\xi)d\xi - \frac{F(0) - G(0)}{2}. \]
Finally

\[y(x, t) = F(x - ct) + G(x + ct)\]

\[= \frac{f(x - ct)}{2} - \frac{1}{2c} \int_{x-ct}^{x} g(\xi) d\xi + \frac{F(0) - G(0)}{2}\]

\[+ \frac{f(x + ct)}{2} + \frac{1}{2c} \int_{0}^{x+ct} g(\xi) d\xi - \frac{F(0) - G(0)}{2}\]

\[= \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi.\]
Consider the 1-d wave equation

\[
\begin{align*}
    c^2 y_{xx} &= y_{tt}, \quad -\infty < x < \infty, \; t > 0, \\
    y(x, 0) &= f(x), \quad y_t(x, 0) = 0, \quad -\infty < x < \infty.
\end{align*}
\]

Since \( g = 0 \), then the solution is given by

\[
y(x, t) = \frac{f(x - ct) + f(x + ct)}{2}.
\]

\( c \) is the **wave speed**; \( x + ct = \text{const.} \) and \( x - ct = \text{const.} \) are the **characteristics**.

**Figure:** Initial pulse, and right- and left- going waves
Example

Consider the 1-d wave equation

\[ c^2 y_{xx} = y_{tt}, \quad -\infty < x < \infty, \ t > 0, \]
\[ y(x, 0) = 0, \ y_t(x, 0) = \delta(x - x_0), \quad -\infty < x < \infty. \]

Since \( f = 0 \), then the solution is given by

\[
y(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \delta(\xi - x_0) d\xi = \frac{1}{2c} \begin{cases} 
1, & x - ct < x_0 < x + ct \\
0, & \text{otherwise}
\end{cases}
\]
d’Alembert’s solution in 3-d

Example
Consider the 3-d wave equation

\[ c^2 \nabla^2 w = w_{tt} \] in polar coordinates

\[ w_{rr} + \frac{2}{r} w_r = \frac{1}{c^2} w_{tt}, \]

where \( w \) only depends on \( r \) and \( t \).

Let \( u = rw \), then

\[ u_r = w + rw_r, \]
\[ u_{rr} = w_r + w_r + rw_{rr} = r(w_{rr} + \frac{2}{r} w_r) = r \frac{1}{c^2} w_{tt} = \frac{1}{c^2} u_{tt}. \]

This gives

\[ c^2 u_{rr} = u_{tt}. \]
From the d’Alembert’s solution to the 1-d wave equation, $u$ has a general solution

$$u = f(r - ct) + g(r + ct).$$

Finally $w$ has a general solution

$$w = \frac{1}{r} f(r - ct) + \frac{1}{r} g(r + ct).$$
The Laplace equation is
\[ \nabla^2 u = 0, \]
and the Poisson equation is
\[ \nabla^2 u = f. \]
Recall the diffusion equation is
\[ \nabla^2 u = u_{tt}, \]
and if \( u \) is a steady-state solution then
\[ \nabla^2 u = 0. \]

**Example**
Solve the following PDE
\[ \nabla^2 u = 0, \quad 0 < x < a, 0 < y < b, \]
\[ u(0, y) = 0, \quad u(a, y) = f(y), \quad 0 < y < b, \]
\[ u(x, 0) = u(x, b) = 0, \quad 0 < x < a. \]
Separation of variables

Seek solutions in the form \( u(x, y) = X(x)Y(y) \). This gives

\[
X''Y + XY'' = 0,
\]

\[
Y(0) = Y(b) = 0.
\]

The first equation gives

\[
\frac{X''}{X} = -\frac{Y''}{Y} = \kappa^2 (\text{constant}),
\]

This gives

\[
X'' - \kappa^2 X = 0,
\]

\[
Y'' + \kappa^2 Y = 0,
\]

and consequently

\[
Y = \begin{cases} 
Jy + K & \kappa = 0, \\
G \cos \kappa y + H \sin \kappa y & \kappa \neq 0
\end{cases}
\]

\[
X = \begin{cases} 
A + Bx & \kappa = 0, \\
C \cosh \kappa x + D \sinh \kappa x & \kappa \neq 0.
\end{cases}
\]