Lecture 17: Analyticity

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Continuity

Let $z_0$ be an interior point in the domain of $f(z)$. We say the limit of $f$, as $z \to z_0$, is $L$

$$\lim_{z \to z_0} f(z) = L,$$

if for any small $\epsilon > 0$ there exists $\delta$ such that

$$|f(z) - L| < \epsilon \quad \text{for any} \quad |z - z_0| < \delta.$$

If $f(z_0) = L$, then we say that $f$ is continuous at $z_0$.

**Example**

$$\lim_{z \to i}(z^2 + iz) = -2.$$
Consider

$$\lim_{z \to i} |(z^2 + iz) + 2| = \lim_{z \to i} |(z - i)(z + 2i)|,$$

For any $\epsilon$, choose $\delta$ such that $\delta < \frac{\epsilon}{4} < 1$, then for $|z - i| < \delta$

$$|(z - i)(z + 2i)| < 4|z - i| < 4\delta < \epsilon.$$

From the definition

$$\lim_{z \to i}(z^2 + iz) = -2.$$
Let $z_0$ be an interior point in the domain of $f(z)$. We define the derivative of $f$ at $z_0$ as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

if the limit exists.

**Example**

Let $f(z) = z^2$, then

$$f'(z) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z} = \lim_{\zeta \to z} (\zeta + z) = 2z.$$
Example

$\frac{1}{z}$ is not differentiable at $z = 0$.

$\overline{z}$ is not differentiable at any point, since

$$\lim_{\zeta \to z} \frac{\zeta - \overline{z}}{\zeta - z}$$

doesn’t exist.
Cauchy-Riemann equations

Let \( z = x + iy = (x, y) \) and consider \( f(z) = u(x, y) + iv(x, y) \). Assume \( f(z) \) is differentiable at point \( z_0 = x_0 + iy_0 = (x_0, y_0) \), then formally

\[
\lim_{(x, y_0) \to (x_0, y_0)} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0) + i[v(x, y_0) - v(x_0, y_0)]}{x - x_0} = u_x(x_0, y_0) + iv_x(x_0, y_0),
\]

\[
\lim_{(x_0, y) \to (x_0, y_0)} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0) + i[v(x_0, y) - v(x_0, y_0)]}{i(y - y_0)} = -iu_y(x_0, y_0) + v_y(x_0, y_0).
\]

By the definition of differentiability, at point \( z_0 = x_0 + iy_0 \)

\[ u_x = v_y, \quad v_x = -u_y. \]

This is the Cauchy-Riemann equations.
Theorem

Let \( z = x + iy = (x, y) \) and consider \( f(z) = u(x, y) + iv(x, y) \). For \( f(z) \) is differentiable at point \( z_0 = x_0 + iy_0 = (x_0, y_0) \),

- it is necessary that the Cauchy-Riemann equations are satisfied at \( z_0 \);
- it is sufficient that the Cauchy-Riemann equations are satisfied at \( z_0 \), and \( u \) and \( v \) be \( C^1 \) in some neighborhood of \( z_0 \).

Suppose that \( f(z) \) is differentiable at point \( z_0 = x_0 + iy_0 = (x_0, y_0) \).

- If \( f \) is differentiable in some neighborhood of \( z_0 \), then \( f \) is analytic at \( z_0 \);
- if \( f \) is not analytic, it is singular;
- if \( f \) is analytic at every point in \( D \), then \( f \) is analytic in \( D \);
- Functions that are analytic everywhere in the complex plane are entire.
Example

Consider $f = |z|^2$, then $f = x^2 + y^2$. This gives that $u = x^2 + y^2$ and $v = 0$. Note that

$$u_x = 2x, \quad u_y = 2y, \quad v_x = v_y = 0.$$  

The Cauchy-Riemann equations hold iff $x = 0$ and $y = 0$. Hence $f$ is only differentiable at $z = 0$ and is analytic nowhere.
Example

Consider \( f(z) = \sin z = \sin x \cosh y + i \cos x \sinh y \). This gives that

\[
\begin{align*}
    u_x &= \cos x \cosh y, & u_y &= \sin x \sinh y, \\
    v_x &= -\sin x \sinh y, & v_y &= \cos x \cosh y.
\end{align*}
\]

The Cauchy-Riemann equations hold everywhere. Then \( f \) is entire.
From the Cauchy-Riemann equations

\[ u_x = v_y, \quad v_x = -u_y, \]

we can have that

\[ \nabla^2 u = u_{xx} + u_{yy} = 0, \quad \nabla^2 v = v_{xx} + v_{yy} = 0. \]

\( u \) and \( v \) are called conjugate harmonic functions.
Conjugate harmonic functions

Let \( u(x, y) = 3xy^2 - x^3 \), then \( u \) satisfies the Laplace equation and is harmonic. Let \( v \) be the conjugate harmonic, then

\[
    v_x = -u_y = -6xy, \quad v_y = u_x = 3y^2 - 3x^2.
\]

Consequently \( v = -3x^2y + A(y) \), and \( -3x^2 + A' = 3y^2 - 3x^2 \). Then \( A = y^3 + c \) and

\[
    v = -3x^2y + y^3 + c.
\]
Let \( z = re^{i\theta} \) and consider \( f(z) = u(r, \theta) + iv(r, \theta) \). Assume \( f(z) \) is differentiable at point \( z_0 = r_0e^{i\theta_0} \), then

\[
\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.
\]

This is the Cauchy-Riemann equations in polar coordinates.

**Example**

Consider the principle value of \( \log z \) defined by \( \log z = \log r + i\theta \), where \( 0 < r < \infty \) and \( -\pi < \theta \leq \pi \). Then \( u(r, \theta) = \log r \) and \( v(r, \theta) = \theta \). This gives that

\[
u_r = 1/r, \quad v_\theta = 1, \quad u_\theta = v_r = 0.
\]

Hence the Cauchy-Riemann equations are satisfied and \( \log z \) is analytic everywhere in the domain defined by the branch cut \( 0 < r < \infty \) and \( -\pi < \theta \leq \pi \).