Lecture 18: Conformal Mapping

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Consider the Laplace equation in a domain $D$,

$$\nabla^2 \psi = \psi_{xx} + \psi_{yy} = 0 \quad \text{in} \quad D.$$

If $D$ is a rectangular or a circular domain (centered at origin), we can use separation of variables. However if $D$ is a general domain, how can we solve the Laplace equation?

If a change of variables is performed,

$$u = u(x,y), \quad v = v(x,y),$$

and $(u, v)$ is the new coordinates in a domain $D'$. We might get a new function $\Psi(u, v) = \psi(x(u, v), y(u, v))$ that is harmonic in $D'$. 
To proceed, assume that this mapping is one-to-one, i.e. the Jacobian is nonzero in $D$,

$$\frac{\partial(u, v)}{\partial(x, y)} = u_x v_y - u_y v_x \neq 0.$$ 

Now from $\psi_x = \Psi_u u_x + \Psi_v v_x$,

$$\psi_{xx} = (\Psi_{uu} u_x + \Psi_{uv} v_x)u_x + \Psi_u u_{xx} + (\Psi_{vu} u_x + \Psi_{vv} v_x)v_x + \Psi_v v_{xx},$$

$$\psi_{yy} = (\Psi_{uu} u_y + \Psi_{uv} v_y)u_y + \Psi_u u_{yy} + (\Psi_{vu} u_y + \Psi_{vv} v_y)v_y + \Psi_v v_{yy},$$

and $\nabla^2 \psi = 0$ becomes

$$(u_x^2 + u_y^2)\Psi_{uu} + (u_x v_x + u_y v_y)(\Psi_{uv} + \Psi_{vu})$$

$$+ (v_x^2 + v_y^2)\Psi_{vv} + (u_{xx} + u_{yy})\Psi_u + (v_{xx} + v_{yy})\Psi_v = 0.$$ 

If $u_x = v_y$ and $u_y = -v_x$, the equation is simplified to

$$(u_x^2 + u_y^2)(\Psi_{uu} + \Psi_{vv}) = 0, \quad \rightarrow \quad \nabla^2 \psi = \Psi_{uu} + \Psi_{vv} = 0.$$ 

$$(u_x^2 + u_y^2 = u_x v_y - u_y v_x = \frac{\partial(u, v)}{\partial(x, y)} \neq 0)$$

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Preservation of Laplace equation

\[ u_x = v_y \text{ and } u_y = -v_x \] are equivalent to that \( f(z) = u(x, y) + iv(x, y) \) is analytic everywhere in a domain \( D \).

\[ u_x^2 + u_y^2 = u_x v_y - u_y v_x = \frac{\partial(u, v)}{\partial(x, y)} \neq 0 \] is equivalent to that \( f'(z) = u_x + iv_x \neq 0 \) everywhere in \( D \).

**Theorem**

Let \( f(z) = u(x, y) + iv(x, y) \) be analytic everywhere in a domain \( D \), and \( f'(z) \neq 0 \) everywhere in \( D \). If \( \psi(x, y) \) is harmonic in \( D \), then \( \Psi(u, v) = \psi(x(u, v), y(u, v)) \) is harmonic in \( D' \), where \( D' \) is the image of \( D \) and the mapping is one-to-one.
Consider the steady-state temperature problem

\[ \nabla^2 \psi = 0 \quad \text{in} \quad D, \]

\[ \psi(0, y) = 10, \]

\[ \psi = 50 \quad \text{on} \quad \partial \Omega_0, \]

where \( \Omega_0 = \{(x, y) : (x - \frac{1}{2})^2 + y^2 < \frac{1}{4}\} \), \( \Omega = \{(x, y) : x > 0\} \) and \( D = \Omega \setminus \overline{\Omega_0} \).
Example

Consider the steady-state temperature problem

\[
\nabla^2 \psi = 0 \text{ in } D,
\]
\[
\psi(0, y) = 10,
\]
\[
\psi = 50 \text{ on } \partial \Omega_0,
\]

where \( \Omega_0 = \{(x, y) : (x - \frac{1}{2})^2 + y^2 < \frac{1}{4}\} \), \( \Omega = \{(x, y) : x > 0\} \) and \( D = \Omega \setminus \overline{\Omega_0} \).

Figure: steady-state temperature problem
Let $D' = \{(u, v) : 0 < u < 1\}$. Assume that $f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ maps $D$ to $D'$ one to one. The boundary conditions give

$$
\Psi(0, v) = 10, \quad \Psi(1, v) = 50.
$$
Hence we can solve for $\Psi$ by separation of variables,

$$
\Psi(u, v) = 10 + 40u.
$$
Finally $\psi(x, y) = \Psi(u(x, y), v(x, y)) = 10 + 40\frac{x}{x^2+y^2}$.
Conformal mapping

A mapping \( w = f(z) \) is conformal at \( z_0 \) if it preserves angles locally, i.e. it preserves the angle between every pair of oriented smooth curves that intersect at \( z_0 \).

Let \( z_j(\tau), j = 1, 2 \) be two oriented smooth curves parameterized by \( \tau \) \((z_j(0) = z_0)\) that intersects at \( z_0 \). Let \( w_j(\tau) = f(z_j(\tau)) \) and \( w_0 = f(z_0) \). Then at \( w_0 \)

\[
dw_j = df dz_j, \quad \rightarrow \quad \arg dw_j = \arg df + \arg dz_j.
\]

This gives that \( \alpha = \arg dz_2 - \arg dz_1 = \arg dw_2 - \arg dw_1 = \beta \).
Conformality

**Theorem**

*If $f(z)$ is analytic, then the mapping $w = f(z)$ is conformal, except at points where $f'(z) = 0$.***