Lecture 19: The Bilinear Transformation

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Möbius transformation

The Möbius transformation (or linear fractional transformation) is a special case of bilinear transformation defined by

\[ w = \frac{az + b}{cz + d} \quad (ad - bc \neq 0). \]

Recall the mapping \( \frac{1}{z} \) that was discussed in the previous lecture, is a Möbius transformation. Now the derivative of \( w \) is given by

\[ w'(z) = \frac{ad - bc}{(cz + d)^2}. \]

Then \( w \) is conformal everywhere except at \( z = -\frac{d}{c} \).

- Point at infinity in the \( w \) plane, denoted as \( \infty \). This occurs when \( z = -\frac{d}{c} \).
- Extended complex plane, is the \( w \) plane augmented by the point at infinity.
- Finite complex plane is referred to the usual complex plane.
We study the Möbius transformation in the extended $z$ and $w$ plane. For completeness we define at $z = \infty$, $\frac{az+b}{cz+d}$ is $\frac{a}{c}$ if $c \neq 0$ and is $\infty$ if $c = 0$.

The Möbius transformation is one-to-one in the extended $z$ and $w$ plane, as demonstrated by

$$z = -\frac{dw - b}{cw - a}.$$
We illustrate the Möbius transformation by $w(z) = 1/z$. Let a circle be defined by

$$(x - a)^2 + (y - b)^2 = c^2,$$

In terms of $z$ it is written as

$$x^2 + y^2 - 2ax - 2by = c^2 - a^2 - b^2 \quad \rightarrow \quad z\overline{z} - Az - \overline{A}\overline{z} = B,$$

where $A = a - ib$ and $B = c^2 - a^2 - b^2$. Now $w = 1/z$ gives

$$\frac{1}{w\overline{w}} - A\frac{1}{w} - \overline{A}\frac{1}{\overline{w}} = B,$$

- If $B \neq 0$,
  $$w\overline{w} + \overline{A}/Bw + A/B\overline{w} = 1/B \quad \text{(circle)}$$

- If $B = 0$, i.e. $c^2 = a^2 + b^2$,
  $$1 = A\overline{w} + \overline{A}w \quad \rightarrow \quad 1 - 2au + 2bv = 0 \quad \text{(line)}$$
Möbius transformation (continued)

- If $B \neq 0$, $w$ maps a circle to a circle.
- If $B = 0$, i.e. $c^2 = a^2 + b^2$, $w$ maps a circle to a straight line.
- $w$ maps a straight line that does not pass the origin, to a circle.

If we consider a straight line as a circle of infinite radius. Then $w$ maps circles to circles.
The Möbius transformation can be decomposed to (1) scaling and rotation, (2) translation, (3) inversion:

\[ w_1 = cz, \quad w_2 = w_1 + d, \quad w_3 = \frac{1}{w_2}, \quad w_4 = \frac{bc - ad}{c}w_3, \quad w = w_4 + \frac{a}{c}, \]

where

- scaling and rotation is: \( Az \),
- translation is: \( z + B \),
- inversion is: \( \frac{1}{z} \).

The Möbius transformation maps circles to circles.
Solve the boundary value problem

\[ \nabla^2 \psi = 0 \quad \text{in} \quad D. \]
Solve the boundary value problem

$$\nabla^2 \psi = 0 \quad \text{in} \quad D.$$ 

$$w(z) = \frac{z-a}{az-1}$$ maps the domain $D$ to $D'$, where $a = \frac{7+2\sqrt{6}}{5}$. Use polar coordinates $(\rho, \phi)$ to solve the Laplace equation in $D'$,

$$\psi_{\rho\rho} + \frac{1}{\rho} \psi_{\rho} + \frac{1}{\rho^2} \psi_{\phi\phi} = 0.$$
Since the domain and boundary conditions are axisymmetric, then solution is

\[ \Psi = A + B \ln \rho, \]

Boundary conditions \( \Psi(R, \phi) = 0 \) and \( \Psi(1, \phi) = 50 \) give

\[ \Psi = 50(1 - \frac{\ln \rho}{\ln R}). \]

Finally

\[ \psi(x, y) = \frac{50}{\ln R} \left( \ln R - \frac{1}{2} \ln \frac{(x - a)^2 + y^2}{(ax - 1)^2 + a^2 y^2} \right). \]