Examples

Solve the boundary value problem

\[ \nabla^2 \psi = 0 \quad \text{in} \quad D. \]

\[ w(z) = z^2 = (x^2 - y^2) + i2xy \] maps the domain \( D \) to \( D' \). The solution in \( D' \) is simply

\[ \Psi(u, v) = 25(v - 2). \]

Hence the solution \( \psi \) in \( D \) is

\[ \psi(x, y) = 25(2xy - 2) = 50(xy - 1). \]

In fact if \( w \) maps \((x, y)\) to \((u, v)\), then \( w \) also maps \((-x, -y)\) to \((u, v)\). The map \( w \) is not one-to-one.
\[ \Psi = (A + B \ln \rho)(C + D\phi) + (E\rho^\kappa + F\rho^{-\kappa})(G \cos \kappa\phi + H \sin \kappa\phi) \]

To have that \(\Psi\) is bounded, \(\Psi = C_1 + C_2\phi\). Boundary conditions \(\Psi(\rho, \pi/4) = 1\) and \(\Psi(\rho, \pi) = 0\) give

\[ \Psi = \frac{4}{3} \left(1 - \frac{\phi}{\pi}\right). \]

Finally

\[ \psi(x, y) = \frac{4}{3} \left(1 - \frac{1}{\pi} \tan^{-1} \frac{2y}{x^2 + y^2 - 1}\right). \]
Examples

Solve the boundary value problem

$$\nabla^2 \psi = 0 \quad \text{in} \quad D.$$ 

This problem can be solved by separation of variables and the solution is represented by an infinite series. To solve it analytically, we use conformal mapping. The map

$$w(z) = -\cos(\pi z) = -\cos(\pi x + i\pi y) = -\cos \pi x \cosh \pi y + i \sin \pi x \sinh \pi y$$

maps the domain $D$ to $D'$. 
The problem in $D'$ can be solved using Fourier transform

$$
\Psi(u, v) = \frac{v}{\pi} \int_{-\infty}^{\infty} \frac{\Psi(u', v)}{(u' - u)^2 + v^2} du' = \frac{v}{\pi} \int_{-1}^{1} \frac{\Psi(u', v)}{(u' - u)^2 + v^2} du' = \cdots
$$

$$
= \frac{100}{\pi} \left[ \tan^{-1}\left(\frac{1 - u}{v}\right) - \tan^{-1}\left(\frac{-1 - u}{v}\right) \right],
$$

where $\tan^{-1}(\cdot)$ belongs to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Finally from $u = -\cos \pi x \cosh \pi y$ and $v = \sin \pi x \sinh \pi y$,

$$
\psi(x, y) = \frac{100}{\pi} \left[ \tan^{-1}\left(\frac{1 + \cos \pi x \cosh \pi y}{\sin \pi x \sinh \pi y}\right) - \tan^{-1}\left(\frac{\cos \pi x \cosh \pi y - 1}{\sin \pi x \sinh \pi y}\right) \right].
$$
More general boundary conditions

- Dirichlet boundary condition:
  \[ \psi(x, y) = \psi(x(u, v), y(u, v)) = \Psi(u, v). \]

- Neumann boundary condition:
  \[ \frac{\partial \psi}{\partial \nu} = \frac{1}{|f'(z)|} \frac{\partial \Psi}{\partial \nu}. \]

Example

The map \( w = z^2 = x^2 - y^2 + i2xy \) maps \( D \) to \( D' \).
To get the Dirichlet boundary condition on $A'B'$, let $xy = 3$, then

$$x(u, \nu) = \pm \sqrt{\frac{u + \sqrt{u^2 + 36}}{2}}, \quad \rightarrow \quad x(u, \nu) = \sqrt{\frac{u + \sqrt{u^2 + 36}}{2}},$$

and this gives

$$\psi(u, 6) = 10e^{-x} = 10e^{-\sqrt{\frac{u + \sqrt{u^2 + 36}}{2}}}. $$
To get the Dirichlet boundary condition on $F'E'$ and $E'C'$, let $x = 0$, then $u = -y^2$ and $v = 0$,

$$y = \pm \sqrt{-u}, \quad \rightarrow, \quad y = \sqrt{-u},$$

and this gives on $F'E'$

$$\Psi(u, 0) = \frac{25}{1 + \sqrt{-u}}.$$

Now on $E'C'$,

$$\Psi(u, 0) = 30.$$
If the boundary condition on $AB$ is changed to $\frac{\partial \psi}{\partial \nu} = 10e^{-x}$. First

$$|f'(z)| = |2z| = 2\sqrt{x^2 + y^2} = \cdots = 2(u^2 + 36)^{1/4}.$$  

Let $xy = 3$, then

$$x(u, v) = \pm \sqrt{\frac{u + \sqrt{u^2 + 36}}{2}}, \quad \rightarrow \quad x(u, v) = \sqrt{\frac{u + \sqrt{u^2 + 36}}{2}},$$

and this gives

$$\frac{\partial \psi}{\partial \nu} = 10 \frac{1}{2(u^2 + 36)^{1/4}} e^{-\sqrt{\frac{u + \sqrt{u^2 + 36}}{2}}}. $$
Applications to fluid dynamics

- Let $q(x, y, z)$ be a fluid velocity field. If the flow is irrotational, i.e. \( \nabla \times q = 0 \), then there exists a scalar potential $\phi$ such that $q = \nabla \phi$.

- If the flow is incompressible ($\nabla \cdot q = 0$), then $\nabla^2 \phi = 0$.

- An irrotational incompressible flow is called a potential flow.

Example (Flow in a corner (in 2-D))

In the $(\xi, \eta)$ plane, $\xi = x^2 - y^2$ and $\eta = 2xy$. 
In the $(\xi, \eta)$ plane, $\Psi = U_0 \zeta$ and hence

$$\psi = U_0 (x^2 - y^2).$$

This gives the velocity flow

$$q = \nabla \phi = 2U_0 (x\hat{i} - y\hat{j}).$$