Lecture 7: Sturm-Liouville Theory

Shixu Meng, Department of Mathematics, University of Michigan

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Consider a linear homogeneous second-order ODE (boundary-value problem)

\[
[p(x)y′]′ + q(x)y + \lambda w(x)y = 0, \quad a < x < b, \\
\alpha y(a) + \beta y′(a) = 0, \\
\gamma y(b) + \delta y′(b) = 0,
\]

where \(a, b\) are finite, where \(p, p′, q, w\) are real and continuous, where \(\alpha, \beta, \gamma, \delta\) are real, and

\[
p(x) > 0, \quad w(x) > 0 \quad \text{on} \quad [a, b].
\]
If there exists a non-trivial solution $y$ to

$$
[p(x)y']' + q(x)y + \lambda w(x)y = 0, \quad a < x < b,
$$

$$
\alpha y(a) + \beta y'(a) = 0,
$$

$$
\gamma y(b) + \delta y'(b) = 0,
$$

for a $\lambda$, then $\lambda$ is called an eigenvalue and the corresponding solution $y$ is called an eigenfunction. Sturm-Liouville eigenvalue problem can generate an orthonormal basis.
Example

Can we generate the basis that has been used in Fourier series?

Example
Consider a particular case

\[ y'' + \lambda y = 0, \quad 0 < x < L, \]
\[ y(0) = 0, \]
\[ y(L) = 0. \]

Find all the eigenvalues \( \lambda \) and the corresponding eigenfunctions.
To incorporate the analysis of the Sturm-Liouville problem, we introduce the inner product between two functions

$$\langle f, g \rangle = \int_{a}^{b} f(x)g(x)w(x)dx.$$ 

When $w(x) = 1$, this is the inner product that has been introduced before.

Why is the inner product weighed by $w(x)$?
Some calculations

Multiply the following equation

\[ [p(x)y']' + q(x)y + \lambda w(x)y = 0, \]

by \( \bar{y} \) and integration over \((a, b)\) yields

\[
\int_a^b [p(x)y']'\bar{y}dx + \int_a^b q(x)y\bar{y}dx + \lambda \int_a^b w(x)y\bar{y}dx = 0.
\]

From integration by parts

\[
\int_a^b p(x)y'\bar{y}'dx - \int_a^b q(x)y\bar{y}dx = \lambda \int_a^b w(x)y\bar{y}dx,
\]

if \( y(a) = y(b) = 0 \) (other boundary conditions are similar).

(Think about a matrix eigenvalue problem \( Ax = \lambda x \))

One can find that \( \lambda \) has to be real-valued.
Sturm-Liouville Theorem

Let $\lambda_n$ and $\phi_n(x)$ denote any eigenvalue and corresponding eigenfunction of the Sturm–Liouville eigenvalue problem (1), respectively.

(a) The eigenvalues are real.

(b) The eigenvalues are simple. That is, to each eigenvalue there corresponds only one linearly independent eigenfunction. Further, there are an infinite number of eigenvalues, and they can be ordered so that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, where $\lambda_n \to \infty$ as $n \to \infty$.

(c) Eigenfunctions corresponding to distinct eigenvalues are orthogonal. That is, if $\lambda_j \neq \lambda_k$, then $\langle \phi_j, \phi_k \rangle = 0$.

(d) Let $f$ and $f'$ be piecewise continuous on $a \leq x \leq b$. If $a_n = \langle f, \phi_n \rangle / \langle \phi_n, \phi_n \rangle$, then the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ converges to $f(x)$ if $f$ is continuous at $x$, and to the mean value $[f(x+) + f(x-)]/2$ if $f$ is discontinuous at $x$, for each point $x$ in the open interval $a < x < b$. 
Example

Discuss the eigenvalues and eigenfunctions for the Sturm-Liouville problems

(1)

\[ y'' + \lambda y = 0, \quad 0 < x < 1, \]
\[ y(0) - 2y'(0) = 0, \]
\[ y(1) = 0. \]

(2)

\[ y'' - 2y' + \lambda y = 0, \quad 0 < x < \pi, \]
\[ y(0) = 0, \]
\[ y(\pi) = 0. \]
Some ideas on the proof of Sturm-Liouville Theorem

Orthogonality; theory of self-adjoint operator;
Periodic and Singular Sturm-Liouville Problems

\[ [p(x)y']' + q(x)y + \lambda w(x)y = 0, \quad a < x < b, \]

- periodic boundary conditions

\[
y(a) = y(b),
\]
\[
y'(a) = y'(b).
\]

- singular case: \( p(a) = 0, p(b) \neq 0 \)

\[
y(a) \text{ bounded,}
\]
\[
\gamma y(b) + \delta y'(b) = 0,
\]

- singular case: \( p(b) = 0, p(a) \neq 0 \)

\[
\alpha y(a) + \beta y'(a) = 0,
\]
\[
y(b) \text{ bounded,}
\]
– singular case: $p(a) = p(b) = 0$

\[ y(a) \text{ bounded,} \]
\[ y(b) \text{ bounded.} \]

1. The eigenvalues are real.

2. Eigenfunctions corresponding to distinct eigenvalues are orthogonal.
Example

Discuss the periodic Sturm-Liouville problem

\[ y'' + \lambda y = 0, \quad -L < x < L, \]
\[ y(-L) = y(L), \]
\[ y'(-L) = y'(L). \]
Example
Discuss the singular Sturm-Liouville problem

\[(xy')' + \lambda xy = 0, \quad 0 < x < L,\]
\[y(0) \text{ bounded,}\]
\[y(L) = 0.\]
Example
Discuss the singular Sturm-Liouville problem

\[(1 - x^2) y'' - 2xy' + \lambda y = 0, \quad -1 < x < 1,\]
\[y(-1) \text{ bounded,}\]
\[y(1) \text{ bounded.}\]