Lecture 22: Cauchy Theorem and Fundamental Theorem of Complex Integral Calculus

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Recall the complex integral is defined by

\[ I = \int_C f(z)dz, \]

where \( C \) is a given path of integration or contour. Let \( C \) be closed and denote the enclosed region by \( D \). Recall further that to evaluate \( \int_C f(z)dz \), consider

\[ \int_C f(z)dz = \int_C (u + iv)(x + iy) = \int_C (udx - vdy) + i \int_C (vdx + udy). \]

- If \( f \) is analytic in \( D \), then the Cauchy-Riemann equations give \( u_x = v_y, \quad v_x = -u_y \).

- Recall the Green's theorem

\[ \int_C (Pdx + Qdy) = \int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}). \]

- \( \int_C f(z)dz = 0. \)
Cauchy’s Theorem

Theorem

If \( f(z) \) is analytic in a simply connected domain \( D \), then

\[
\int_C f(z) \, dz = 0
\]

for every piecewise smooth simple closed curve \( C \) in \( D \).
Example

Evaluate \( \int_C \frac{1}{z^2 - 5z + 6} \, dz \), where \( C \) is a contour as shown in the following Figure.

Since \( f(z) = \frac{1}{z^2 - 5z + 6} = \frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2} \). Hence \( f \) is analytic in \( D \). From Cauchy’s Theorem

\[
\int_C \frac{1}{z^2 - 5z + 6} \, dz = 0.
\]
**Path independence**

**Theorem**

If $f(z)$ is analytic in a simply connected domain $D$, $P$ and $Q$ are two given points in $D$. Let $C_j, j = 1, 2$ be any simple piecewise smooth path connecting $P$ and $Q$. Then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$ 

This can be derived from Cauchy’s Theorem, since $C_1 + (-C_2)$ is a closed path in $D$ and

$$\int_{C_1 + (-C_2)} f(z)dz = 0.$$
Corollary

Let $D_j, j = 1, 2$ be simply connected domain enclosed by $C_j, j = 1, 2$ and $D_2 \subset D_1$. Let the orientation of $C_1$ and $C_2$ be same, and $f(z)$ be analytic in $D_2 \setminus D_1$. Then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

This can be derived from Cauchy's Theorem,

$$\int_{C_1+(-C_2)} f(z)dz = 0 \iff \int_{C_1+(-C_2)} f(z)dz + \int_{C'+(-C')} f(z)dz = \int_{C_1+C'+(-C_2)+(-C')} f(z)dz = 0$$
Example
Evaluate \( \int_{C} (z - a)^{n} \, dz \), where \( C \) is a contour as shown in the following Figure.

From Cauchy’s Theorem and path independence,
\[
\int_{C} (z - a)^{n} \, dz = \int_{C'} (z - a)^{n} \, dz.
\]

When \( n + 1 \neq 0 \), the integral is simply zero,
\[
\int_{C} (z - a)^{n} \, dz = 0.
\]

When \( n + 1 = 0 \),
\[
\int_{C} f(z) \, dz = 2\pi i.
\]
Example
Evaluate $\int_C \frac{1}{z^2(z-2)(z-4)} \, dz$, where $C$ is a contour as shown in the following Figure.

Since

$$\frac{1}{z^2(z-2)(z-4)} = -\frac{1}{8} \frac{1}{z-2} + \frac{1}{32} \frac{1}{z-4} + \frac{3}{32} \frac{1}{z} + \frac{1}{8} \frac{1}{z^2},$$

then from Cauchy’s Theorem and path independence,

$$\int_C \frac{1}{z^2(z-2)(z-4)} \, dz = \int_C \left[ -\frac{1}{8} \frac{1}{z-2} + \frac{3}{32} \frac{1}{z} + \frac{1}{8} \frac{1}{z^2} \right] \, dz$$

$$= -\frac{1}{8} (2\pi i) + \frac{3}{32} (2\pi i) = -\frac{1}{16} \pi i.$$
Fundamental theorem of complex integral calculus

Let $z_0$ be a fixed point in a simply connected domain $D$, and $f(z)$ be analytic in $D$. Consider

$$\int_{z_0}^{z} f(\zeta) d\zeta,$$

which is well-defined (because of path independence) and interpreted as the integral of $f(z)$ along any path connecting $z_0$ and $z$. If we denote $G(z) = \int_{z_0}^{z} f(\zeta) d\zeta$, then the derivative of $G$ is

$$G'(z) = f(z).$$

We call any function $F(z)$ such that $F'(z) = f(z)$ the primitive of $f$. Now let $F$ be a primitive of $f$,

$$\int_{z_0}^{z} f(\zeta) d\zeta = \int_{z_0}^{z} F'(\zeta) d\zeta = F(z) - F(z_0).$$
Theorem (Fundamental theorem of complex integral calculus)

Let \( f(z) \) be analytic in a simply connected domain \( D \), and \( z_0 \) be a fixed point. Then

- \( G(z) = \int_{z_0}^{z} f(\zeta) \, d\zeta \) is analytic in \( D \) and \( G'(z) = f(z) \).
- If \( F \) is a primitive of \( f \), then

\[
\int_{z_0}^{z} f(\zeta) \, d\zeta = F(z) - F(z_0).
\]
Example
Evaluate $\int_C z \, dz$, where $C$ is a contour as shown in the following Figure.

Since $f(z) = z$, then a primitive of $f$ is $F(z) = \frac{1}{2} z^2$. This gives that

$$\int_C f(z) \, dz = \int_{2i}^{-2i} F'(z) \, dz = \frac{1}{2} z^2 \bigg|^{{-2i}} _{2i} = 0.$$