ANATOLE KATOK'S WORK ON COHOMOLOGY AND GEOMETRIC RIGIDITY

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ABSTRACT. We will provide a brief introduction to one major part of Anatole Katok's work, concerning geometric rigidity. This includes how regularity properties of dynamically defined foliations lead to rigidity, and the fascinating story of smooth rigidity of higher rank actions by abelian and semisimple groups and their lattices. We will also some of the further progress made, and discuss some open problems.

1. INTRODUCTION

My association with Anatole Katok began in 1980 when he visited University of Warwick in England. At the time, I was a graduate student there. This was also precisely when Robert Zimmer developed his superrigidity theorem for cocycles of actions of semisimple Lie groups of higher (real) rank. Katok was interested in running a year long program at University of Maryland studying these exciting developments. I happened to have some acquaintance with the theory. The rest was history as they say, and I assisted with this program at Maryland, much to my advantage at least. To my knowledge, this is more or less when Katok's serious involvement with rigidity began. Of course he was well aware of the amazing works by Furstenberg, Mostow and Margulis. But he had not worked on rigidity issues before even though Margulis' techniques in particular involved ideas from ergodic theory, in particular Lyapunov exponents and Oseledets splitting.

This was also the time when rapid progress occurred in the theory of Anosov and more general hyperbolic flows and applications to Riemannian geometry, especially in negative and non-positive sectional curvature. Anosov's work itself laid the groundwork, giving long sought for proofs of ergodicity for the geodesic flow of negatively curved closed manifolds with respect to the Liouville measure, the natural invariant volume on the unit tangent bundle. In negative curvature, Margulis famously found the asymptotic for the number of closed geodesics of certain lengths. R. Bowen described the equidistribution of the closed geodesics as the unique measure of maximal entropy. Pesin theory allowed generalizations to non-uniformly hyperbolic systems, such as geodesic flows in rank 1 manifolds of nonpositive curvature, using the ideas of Lyapunov exponents and Oseledets decompositions, even though ergodicity with respect to the Liouville measure still remains open. In some situations this allowed for counting closed orbits.
in the context of flows of a 3-dimensional manifold preserving a non-uniformly hyperbolic measure as was studied by Katok in his breakthrough paper \cite{30}. This generalized the work of Bowen and Margulis \cite{41, 84}. 

Katok also worked with S. Hurder on regularity problems of dynamically defined foliations at the time which resulted in their remarkable work on characteristic classes and foliations. For low dimensions, they still provide the best results in terms of optimal regularity. Naturally this led to similar questions in higher dimension, and in particular to the works of Katok with Flaminio and later Feres, These topics will form the first part of this overview.

Around this time, our understanding of rigidity of manifolds of nonpositive curvature of higher rank developed quickly, based on foundational work by Ballmann, Brin, Eberlein and this author \cite{36}. Two quite different solutions were found by Ballmann and Burns-Spatzier \cite{35, 46}. The second was achieved in a remarkable year long program at MSRI from 1983 to 1984 \cite{31} where Katok served as a main organizer. Both methods used dynamics crucially. And maybe not surprisingly, the idea arose that vice versa higher rank hyperbolic abelian actions might enjoy their own special rigidity properties. Indeed I posed this as a somewhat naive question at a problem session at the end of the year at MSRI \cite{45}. This immediately drew Katok’s attention and we had an excited exchange afterwards. While nothing happened for some time, the ideas stayed and eventually became crucial in the Zimmer program and the study of rigidity properties of hyperbolic actions of higher rank abelian groups, and much more. Amazingly, one other major development of this special year, the Ledrappier-Young work on entropy later became a crucial tool in work on higher rank rigidity in dynamics. I will report on Katok’s interaction with this branch of dynamics quite extensively below, organized by topics. The closely related subject of measure rigidity will be discussed in the next chapter by Federico Rodriguez Hertz \cite{28}.

2. Regularity of Foliations, Characteristic Classes and Rigidity

As it will be basic to every aspect of this introduction, let us first recall the notion of Anosov diffeomorphisms and flows:

*Let $M$ be a compact manifold with some ambient Riemannian metric $\|\|$. Then a diffeomorphism $f : M \to M$ is called Anosov if there is a continuous splitting of the tangent bundle $TM = E^s \oplus E^u$ and constants $C > 0$ and $\lambda > 0$ such that for all $n > 0$, $v \in E^s$ and $w \in E^u$\n
$$\|df^n(v)\| \leq Ce^{-n\lambda}\|v\| \text{ and } \|df^{-n}(w)\| \leq Ce^{-n\lambda}\|w\|.$$*

These are very common yet fascinating dynamical systems with many representatives. In particular linear automorphisms of tori without eigenvalues of modulus 1 are examples, as are geodesic flows of closed manifolds of negative sectional curvature \cite{82}. Their stable and unstable distributions are integrable and give rise to stable and unstable foliations. However, while the individual leaves are smooth immersed submanifolds, the transverse regularity is in general
only Hölder. In special situations, we get $C^1$-transverse regularity, for example for geodesic flows of closed surfaces of negative curvature or higher dimensional manifolds with metrics those sectional curvature is strictly $\frac{1}{4}$-pinched sectional curvatures \cite{73,72}. This was studied in detail in terms of bunching by B. Hasselblatt in his thesis, directed by Katok, and subsequent publications \cite{68,69,70}. In particular, Hasselblatt found examples of metrics with curvature pinched close to $\frac{1}{4}$ for which the stable foliations do not have transversal $C^1$-regularity \cite{68}.

Closely related is the problem of optimal regularity. One sees easily that manifolds of constant negative curvature, more generally locally symmetric manifolds and even more generally stable foliations of homogeneous systems are always $C^\infty$, even real analytic. One could dream that only such systems have smooth stable foliations. Maybe even $C^2$ transversal regularity could suffice? For surfaces this problem was tackled optimally by Hurder and Katok in \cite{2}. In fact, they study this under a refined regularity condition, $C^{1+\text{Zygmund}}$. Short of defining this let us just say that a Zygmund function has modulus of continuity $O(s \log s)$, and that any Hölder function is Zygmund. Let us state just one striking result of their work \cite{2} (we note that the version of the paper in this volume is not the published one but rather a much more extensive preprint version):

**Suppose that a $C^\infty$ volume preserving Anosov flow of a closed 3-manifold has weak stable and weak unstable foliations with $C^{1+\text{Zygmund}}$ transversal regularity. Then they are $C^\infty$ foliations. In addition, if the flow is a geodesic flows of a closed surface $M$, then $M$ has constant curvature.**

The proofs of this result are rather involved and pass through a construction and evaluation of a Godbillon-Vey invariant for foliations of low regularity. This followed earlier work of Hurder and Katok \cite{11} in which they defined measurable Godbillon-Vey classes for $C^2$-foliations and proved their vanishing for amenable foliations. Furthermore, stable foliations of uniformly hyperbolic systems are always amenable by work of Bowen (in fact they have polynomial growth so that standard averaging techniques give amenability).

Not nearly as much is understood in higher dimension. All known results either presume the stable foliations to be $C^\infty$ or minimally to have some high finite regularity. Assume for simplicity that they are $C^\infty$ (transversely). Then Kanai proved in \cite{80} in 1988 under a strong pinching assumption on the curvature, $-\frac{9}{4} < K < -1$, that such a metric had to have constant curvature. Kanai famously introduced a new tool, now called the Kanai connection. With Feres \cite{3}, Katok formulated and proved various results for the vanishing of tensor fields using Lyapunov criteria in terms of Lyapunov exponents. Roughly these describe the exponential growth rates of vectors under the flow (we refer to volume 1 for details on Lyapunov exponents). This allowed them to improve Kanai’s curvature condition to $\frac{1}{4}$ pinching. In a later paper with Feres \cite{4}, Katok investigated the case of geodesic flows of negatively curved 3-manifolds, proving a dichotomy: either the curvature is constant or the Lyapunov exponents of the flow must be resonant. In fact they have to be multiples by 2 (later generalized by Feres to odd
dimensional manifolds). This can be viewed as a precursor to much of the later work on normal forms, cf. [8]. Note that the factor 2 corresponds to a resonance which makes normal forms non-linear. With Flaminio in [5], Katok proved similar results for symplectic Anosov diffeomorphisms on low dimensional tori.

The higher dimensional case of volume preserving Anosov flows with $C^\infty$ stable distributions was eventually solved by Benoist, Foulon and Labourie in [37] for the case of contact Anosov flows. This applied in particular to geodesic flows in negative curvature. They showed that such flows have to be $C^\infty$-conjugate to a geodesic flow of a closed locally symmetric space of negative curvature. To achieve this, they adapted Kanai’s construction of a $C^\infty$ connection invariant under the flow. This gave them dense orbits for the ambient local group of this connection preserving automorphisms, by ergodicity of the Anosov flow, which in turn allowed them to apply a principal result of Gromov’s on rigid geometric structures [65]. Later work by Besson, Courtois and Gallot completed the picture via an application of their entropy rigidity theorem: They in fact showed that the underlying Riemannian metrics have to be locally symmetric. For contact Anosov flows, there are also certain time changes of geodesic flows of locally symmetric spaces with this property. Finally, Benoist and Labourie [38] also established similar results for symplectic Anosov diffeomorphisms. Eventually other major conjectures got a much better understanding from these new techniques. Case in point are Lichnerowicz’ conjecture on harmonic and more generally asymptotically Riemannian manifolds: In negative curvature, Foulon and Labourie proved that such manifolds have smooth stable foliations [60] and hence are locally symmetric. This was proven more generally for compact non-positively curved manifolds by A. Zimmer [101].

The general problem when an Anosov diffeomorphism or flow has $C^\infty$ stable foliations remains outstanding, and it is unclear to what extent the symplectic/contact structures are truly needed. Note however that smooth time changes of geodesic flows of locally symmetric spaces with $C^\infty$ stable and unstable foliations again have this property (cf. e.g. [49, Proposition 5.1]). For diffeomorphisms, one can construct more examples via skew products of Anosov diffeomorphisms.

Besides stable and unstable foliations, one can also speculate about other dynamical objects and their rigidity properties assuming some smoothness. As an example, just suppose that the “slow distribution” of an Anosov geodesic flow is $C^\infty$. Can we conclude the underlying Riemannian metric is locally symmetric? Here we call a distribution tangent to the unstable manifold “slow” if it comprises a sequence of Oseledets spaces organized in increasing order of associated Lyapunov exponents. This has connections with classifying negatively curved closed manifolds of higher hyperbolic rank, i.e. when every geodesic has a parallel field that makes extremal curvature -1 with the geodesic. If the sectional curvature $K$ is bounded above by -1, this is a major result of Hamenstädt [65].
For $K \geq -1$ this is still a largely unsolved problem \cite{48, 49}. For recent progress we refer to \cite{48, 49}.

Furthermore, there are close connections of these issues with another aspect of Katok’s work, that of entropy and rigidity of closed geodesics, discussed by Ya. Pesin in volume I. Recall the notions of topological entropy and measure theoretic entropy from there. Katok famously conjectured for geodesic flows of manifolds of negative curvature that the natural invariant volume (Liouville measure) has maximal entropy precisely when the underlying Riemannian metrics is locally symmetric \cite{30, 45}. For the case of surfaces, Katok proved this in \cite{30}. Since then, much progress on various entropy conjectures has been made. In particular, the behavior of topological entropy at the locally symmetric loci is well understood by works of Katok with Knieper, Pollicott and Weiss \cite{32, 33}. After proving smoothness of topological entropy in terms of the metric, they provide explicit formulas for the derivative and show criticality at the symmetric space locus. For metrics of constant negative curvature, Flaminio manages to compare topological and metric entropies near constant curvature, and obtains a local version of the entropy conjecture there \cite{58}. The general case of locally symmetric spaces though is outstanding. To be clear, the general case of Katok’s entropy conjecture is still wide open and seems to require novel ideas.

In summary, Katok’s geometric rigidity program had a large impact already on the Riemannian geometric side.

### 3. Early Work on the Zimmer Program

As mentioned above, Zimmer’s superrigidity theorem for cocycles provided Katok with one point of entry to rigidity problems in dynamics. First recall the notion of cocycle. Given an action of $G$ on a measure space $X$ and another group $H$, we call a map $\alpha : G \times X \to H$ a cocycle if $\alpha(ab, x) = \alpha(a, bx) \cdot \alpha(b, x)$. Homomorphisms provide natural examples, and are called constant cocycles.

At the time of Zimmer’s theorem, one main interest came from measurable orbit equivalence. In general this is a very soft notion as any two measure preserving invertible transformations are measurably orbit equivalent. This is known as Dye’s theorem \cite{51} and was generalized to both discrete and continuous amenable group actions by Connes, Feldman and Weiss \cite{47}. As a consequence of superrigidity for cocycles however, Zimmer showed that two non-isomorphic higher rank semisimple Lie groups cannot have measure preserving actions which are orbit equivalent \cite{102, 103}. This later even found applications to descriptive set theory by Adams and Kechris \cite{34}.

Zimmer’s superrigidity for cocycles also naturally applies to the derivative cocycle $\alpha$ of an action of a higher rank semisimple Lie group $G$ or lattice $\Gamma$ in $G$. Simply fix a measurable framing $\tau$ of the frame bundle of $M$. Then for $g \in G$ the derivative $dg$ determines a cocycle via $dg(\tau(x)) = \alpha(g, x)\tau(gx)$. If the action preserves a probability measure $\mu$, Zimmer’s theorem implies that after a
measurable change of the framing $\tau, \alpha$ can be written as a product of a homomorphism $\rho : G \to GL(m, \mathbb{R})$ where $m = \dim M$ and a cocycle $\kappa$ into a compact group which commutes with the image of $\rho$. One calls $\rho$ the superrigidity homomorphism. Right away this determines the Lyapunov exponents and measure theoretic entropy of $\mu$ in terms of $\rho$.

This is very strong information indeed, and caused Zimmer to speculate what we can say about smooth actions on compact manifolds of such groups, searching for a non-linear version of Margulis’ superrigidity theorem. This started the so-called Zimmer program, and in particular questions about existence of low dimensional actions and connections with dynamics. Indeed, the so-called Zimmer conjecture says that sufficiently low-dimensional actions of such $\Gamma$ should only arise via mapping $\Gamma$ to a finite group $F$, and letting $F$ act. Low dimensional simply means that the ambient group $G$ does not have non-trivial representations in dimensions up to $\dim M$. Hence the derivative cocycle takes values in a compact group, and one immediately gets a measurable invariant Riemannian metric. That this metric is smooth was only recently achieved by Brown, Fisher and Hurtado in celebrated works [42, 43].

Zimmer also called for classification of actions of $SL(n, \mathbb{Z}), n > 2$, on the $n$-torus in his ICM address [104] in 1986. His student J. Lewis established infinitesimal rigidity of the standard action of $SL(n, \mathbb{Z})$ in his thesis [83]. Shortly thereafter Hurder proved deformation rigidity in [74], strongly using higher rank abelian groups contained in $SL(n, \mathbb{Z})$ for $n > 2$. Katok in joint work with Lewis [6, 11], and also Zimmer [12] then almost immediately proved local rigidity of such actions (cf. Section 4 below for definitions). The main point is simply that actions of higher rank abelian groups are already quite rigid. This can then be used to deduce rigidity of the ambient $\Gamma$- or $G$-action. Note that various devices from dynamics, the Oseledets splitting in particular, can be realized simultaneously for all elements of an abelian group. This is essentially like joint diagonalization of commuting matrices. These ideas have been a driving force of the subject. These papers also introduced a normal forms tool that proved invaluable for future developments.

In [11], Katok and Lewis furthermore use a “blow-up” construction of fixed points on tori and find the first examples for:

*There are smooth ergodic volume preserving actions of higher rank lattices that are not $C^1$-conjugate to an algebraic action. These examples can be made real analytic.*

Inspired by this construction, Benveniste in his Chicago thesis found many more examples of similar flavor [39].

Finally and quite a bit later, with Rodriguez Hertz in [25], Katok studied real analytic actions of $SL(n, \mathbb{Z})$ on the $n$-torus $T^n$. Suppose the induced action on homology is standard and that $SL(n, \mathbb{Z})$ preserves a probability measure not supported on a (proper) ball. Then the action has to be smoothly conjugate to the
linear action on $T^n$ on a dense open $SL(n, \mathbb{Z})$ invariant set in $T^n$. This is best possible as the examples of Katok and Lewis show. On the other hand, in this most basic case, it confirms a conjecture of Zimmer that all volume preserving actions of lattices should be pieced together from affine algebraic pieces (cf. Margulis’ list of problems for the new century [85]).

4. LOCAL RIGIDITY

Let us back up a bit and define local rigidity. Suppose $\Gamma$ is a finitely generated group with finite generating set $S$, and let $\kappa$ be an action of $\Gamma$ on a closed manifold $M$ by $C^\infty$-diffeomorphisms. Consider another action $\kappa'$ of $\Gamma$ on $M$, and suppose that $\kappa'(s)$ is $\delta$-close to $\kappa(s)$ in the $C^k$-topology for some $k \geq 1$ and all $s \in S$. We call $\kappa$ locally rigid if $\kappa'$ is $C^\infty$ conjugate to $\kappa$ for some $\delta > 0$. More precisely, one speaks of $C^{k,\infty}$-local rigidity.

The most common case is for $k = 1$. However, if the action is only partially hyperbolic, one usually has to consider $k > 1$, typically to apply KAM-theory. For Lie groups $G$, one replaces finite generating sets by compact ones. Also one has to allow for perturbed actions obtained by conjugating $\kappa$ by an automorphism of $G$. One then speaks of local rigidity up to an automorphism. This is particularly important if $G = \mathbb{R}^k$ where this corresponds to a “constant time change” of the action.

The various results of Hurder, Katok, Lewis and Zimmer certainly raised hope for a rather general picture for rigidity of actions of semisimple groups and their lattices when they contain a higher rank abelian subgroup $A$ that acts uniformly hyperbolically. It turned out that the case of Lie group actions was actually easier to discuss. We suppose that the action preserves a probability measure $\mu$. Then we get Lyapunov exponents for every element $a \in A$. In fact, we can refine the Oseledets decomposition in Pesin theory to get a jointly $A$-invariant measurable decomposition of the tangent bundle $TM = \oplus E^a_\lambda$ of $M$ such that $v \in E^a_\lambda$ has Lyapunov exponent $\lambda(a)$ where the $\lambda$ are a finite collection of linear functionals on $A$. We refer to [44] for a detailed exposition.

Now let us first discuss local rigidity of higher rank abelian actions. This will be of independent interest but also leads to rigidity results for semisimple groups. Let us expand on the notion of Anosov actions to include $A = \mathbb{R}^k$ actions. As in Definition 2, we ask for an element $a \in A$ and a continuous splitting of the tangent bundle into subspaces $E^a_\lambda$ and $E^u_\lambda$ and the tangent space of the $\mathbb{R}^k$-orbit such that stable (unstable) spaces get exponentially contracted by $a^n, n \to \infty$ ($n \to -\infty$ respectively) with precise rates as in Definition 2. In this case, we call $a \in \mathbb{R}^k$ an Anosov or uniformly hyperbolic element, and the action Anosov. There are many Anosov actions of $\mathbb{Z}^k$ or $\mathbb{R}^k$, coming from algebraic or homogeneous systems. For example, we can consider the action by the diagonal subgroup on $SL(n, \mathbb{R})/A$ where $A$ is a uniform lattice in $SL(n, \mathbb{R})$.

My joint work with Katok on local rigidity dates back to the early 1990s and, incidentally, another very successful stay at MSRI. It concerned both actions of
and \( \mathbb{Z}^k \) on nilmanifolds and spaces like \( SL(n, \mathbb{R})/\Lambda \) as well as skew products of such. We called the first *Weyl chamber flows*, and the last *twisted Weyl chamber flows*, and more generally define the class of *standard examples* via a recursive procedure. We then have the main result of [16]:

*The standard higher rank abelian homogeneous action are \( C^{1,\infty} \) local rigid up to automorphisms.*

I will be discussing various ingredients entering this result below. However, let us first mention some related work and further developments.

First, we derive \( C^{1,\infty} \) local rigidity of higher rank lattice actions on tori and more generally nilmanifolds by automorphisms. About ten years later, Margulis in works first with Qian and ultimately Fisher obtained the ultimate results for local rigidity of higher rank \( G \) or \( \Gamma \) affine actions [54]. Their approach is different, writing the perturbation in terms of a cocycle over the original action, and using superrigidity techniques directly on that. They still use the presence of higher rank abelian subgroups though.

Our work also has repercussions for “projective” actions of lattices. More precisely, let \( G \) be a semisimple Lie group, \( \Gamma \) a uniform lattice and \( P \) a parabolic subgroup of \( G \). We call the action of any subgroup of \( G \) by left translations on \( G/P \) a *projective* action. In [16], Katok and Spatzier use an idea of Ghys to obtain a dual statement:

*The projective action of a uniform lattice \( \Gamma \subset G \) of real rank at least 2 on \( G/P \) is \( C^{1,\infty} \) locally rigid.*

Intriguingly, Kanai had proved similar results for \( G = SL(n, \mathbb{R}) \) using cohomology vanishing of suitable foliations and a harmonic representation of tangential forms [81] under somewhat stronger assumptions on the perturbation.

The idea is to build a perturbation of the Weyl chamber flow from a perturbed projective action, and use the local rigidity of the Weyl chamber flow. To my knowledge, the case of projective actions of non-uniform lattice is still open.

### 5. Conjectures and State of the Art

The results discussed give strong evidence for the following conjecture of Katok and Spatzier [45, 71]. As it is obviously false for products of Anosov diffeomorphism or flows, we need to formulate an irreducibility condition first. Let \( \rho_0 \) be a \( C^\infty \) action of \( \mathbb{Z}^d \) or \( \mathbb{R}^d \) on a closed manifold \( M \). A factor action on a closed manifold \( N \) with \( C^\infty \) factor map \( \pi : M \to N \) is called a *rank one factor* if a subgroup of rank \( d-1 \) acts trivially on \( N \). If no such factor exist, we say that the action on \( M \) does not have a rank 1 factor.

**Conjecture:** *All higher rank \( C^\infty \) Anosov actions on any compact manifold without rank one factors are \( C^\infty \) conjugate to an algebraic action after passing to finite covers.*
Non-product examples of actions with rank 1 factors exist, and were first constructed by Starkov (cf. [16]). If an action has a rank 1 factor, one can perturb the factor to get non-algebraic perturbations. So not even local rigidity can hold.

This conjecture is similar to the classical conjecture by Anosov and Smale that Anosov diffeomorphisms are topologically conjugate to an automorphism of an infra-nilmanifold [94]. However, the higher rank and irreducibility assumptions claim much stronger conclusions: in particular, the conjugacy is supposed to be smooth. For single Anosov diffeomorphisms, this is impossible as one can locally change derivatives at fixed points - an obstacle to being even \( C^1 \)-conjugate. Furthermore, Farrell and Jones and later Farrell and Gogolev gave examples of Anosov diffeomorphisms on tori with exotic smooth structures [52, 53]. Thus, for Anosov diffeomorphisms not even the differentiable structure of the underlying manifold has to be standard (although after passing to a finite cover, it becomes standard). More importantly, our conjecture applies equally well to \( \mathbb{Z}^k \) and \( \mathbb{R}^k \)-actions. There is no reasonable conjecture for classifying Anosov flows. Indeed there are many such, coming from geodesic flows of the rich class of closed manifolds of negative curvature. Handel and Thurston constructed even more using a cutting and pasting procedure [67]. Also Anosov flows need not be transitive, e.g. the Franks-Williams examples [59]. Whether such examples (without rank 1 factors) exist for Anosov actions of higher rank abelian groups is not known. We remark that the structure theory of topologically transitive higher rank actions has been studied extensively by Spatzier and Vinhage [95].

Significant progress on the Katok-Spatzier conjecture has been achieved during the last decade. For higher rank Anosov \( \mathbb{Z}^k \) actions on tori and nilmanifolds, Rodriguez Hertz and Wang [92] have proved the ultimate result: global rigidity assuming only one Anosov element, and that its linearization does not have affine rank one factors. It followed earlier work by Fisher, Kalinin and Spatzier [57] for totally Anosov actions, i.e. Anosov actions with a dense set of Anosov elements. As one needs to know that the underlying manifold is a nilmanifold, this is really a global rigidity theorem. Finally, in a similar vein, Spatzier and Yang classified nontrivially commuting expanding maps in [96]. This used results of Gromov and Shub [64, 93] that expanding maps are \( C^0 \) conjugate to endomorphisms of nilmanifolds (up to finite cover). We remark that the basic construction of the “past” space as a \( p \)-adic extension of the real manifold was inspired by treatment of the \( \times 2, \times 3 \) measure rigidity result of Katok and myself in [27]. Finally, note that a positive resolution of the conjecture by Anosov and Smale would force the underlying manifold to have a closed nilmanifold as a finite cover. Combined with the results above, this would automatically prove the Katok-Spatzier conjecture for higher rank \( \mathbb{Z}^k \) actions.

The case of \( \mathbb{R}^k \) actions or when the underlying manifold is not known is not as well understood. In particular, there are currently no arguments when the action is not totally Anosov. Katok in fact saw this as the most difficult part of the
conjecture. There has been good progress in other aspects though. Call a totally Anosov action \textit{Cartan} if the maximal nontrivial intersections of stable manifolds of Anosov elements are 1-dimensional. In that case, a result of Kalinin and Spatzier \cite{79} provides Hölder norms on these intersections which scale precisely according to a Lyapunov exponent under the action. This is really a cocycle rigidity result, similar to Livsic’ theorem but in a particular partially hyperbolic situation. They used this to classify such actions of rank 3 or higher groups under additional assumptions. It was inspired by Goetze and Spatzier’s classification of Cartan actions of lattices in \cite{61}. Later works by Kalinin, Sadovskaya, and Damjanovic and Xu classify Anosov actions under joint integrability or non-resonance conditions \cite{77, 76, 50}. Recently, Vinhage and Spatzier gave a classification on arbitrary closed manifolds under fairly general assumptions \cite{95}. We use free products of Lie groups, a tool introduced by Vinhage in his thesis. Work is under way to extend these ideas to Anosov actions of higher rank semisimple groups.

Naturally, one can ask for various generalizations of the main conjecture, in particular for non-uniformly hyperbolic or partially hyperbolic actions. At this point though, these are mere dreams.

6. Some Ideas in the Proof of Local Rigidity

Let us return to the local rigidity result of Katok and Spatzier for standard Anosov actions, and explain a few of the main ideas introduced. We use the notation from Section 4, and let $\lambda$ be a Lyapunov functional on $\mathbb{R}^k$. The key difference between rank one and higher rank actions is that $\lambda$ will vanish on a non-trivial hyperplane in $\mathbb{R}^k$ for $k \geq 1$ while in rank 1 we just a get a number. Moving with an element $a \in \ker \lambda$ in the homogenous space $G/\Gamma$ gives a motion which exponentially distorts all directions except the $E_{\lambda'}$ for $\lambda'$ proportional to $\lambda$. This allows us to take limits along the foliations $W_\lambda$ corresponding to $E_{\lambda'}$.

Next we can use structural stability of Anosov actions to get a Hölder orbit equivalence $\phi$ close to the identity between original and perturbed action, transversely unique. The goal is to prove smoothness of $\phi$. The idea is to prove smoothness of $\phi$ along the $W_\lambda$. Then a Journè type argument yields smoothness of the orbit equivalence.

Now $\phi$ will intertwine the limit actions on the $W_\lambda$ discussed above. To force smoothness, it suffice to show that the new actions on the new $W_\lambda$ are transitive and smooth. The first is done using ergodicity of the $\ker \lambda$ subgroups. Smoothness required generalizing the normal forms argument from Katok and Lewis \cite{6} from dimension 1 to higher dimension. Normal forms will guarantee local coordinates in which the transformations and their centralizers can be written as polynomial maps of bounded degree, or more precisely in terms of a fixed finite dimensional Lie group. Then taking limits in the $C^\infty$ topology is easy: it is just taking limits of elements in the Lie group which are of course still in the Lie
group and act smoothly. This led to an investigation of normal forms discussed in more detail below 8.

We remark that the idea of studying limits of isometries along suitable foliations is also key to the early work of Katok and Spatzier on measure rigidity in [27]. There the limiting transformations leave conditional measures along the same foliations invariant (a.e. with respect to the conditional measure), proving that they are either atomic or Haar measures. We refer to F. Rodriguez Hertz article in this volume for a detailed introduction to measure rigidity, and especially the underlying ideas.

Finally having nailed down the behavior transversal to the orbits, one still has to understand what happens along the orbit. This boils down to understanding “time changes” where time is now $\mathbb{R}^k$ valued. This in turn was solved in an earlier paper by Katok and myself [7] in which we prove cocycle rigidity. Indeed, the time changes determine a cocycle which turns out to be $C^\infty$ conjugate to a constant cocycle. The latter is the automorphism in the local rigidity up to an automorphism claim of the local rigidity theorem. We refer to 7 for more details.

7. COCYCLE RIGIDITY AND HIGHER COHOMOLOGY

Given an action of $\mathbb{R}^k$ on a manifold $M$, we call a map $\alpha : \mathbb{R}^k \times M \to \mathbb{R}$ a cocycle if $\alpha(ab, x) = \alpha(a, bx) + \alpha(b, x)$. This is simply an $\mathbb{R}$-valued cocycle in our earlier discussions on cocycles. The basic result from [7] asserts that $C^\infty$ cocycles over a standard higher rank actions are $C^\infty$-cohomologous to a constant cocycle. This simply means that there is a $C^\infty$ function $f : M \to \mathbb{R}$ such that $\alpha(a, x) = f(ax) - f(x)$. In terms of time changes, this allows us to straighten time to become linear from point of view of the given action. Later this result was also proved for Hölder cocycles, for a Hölder coboundary.

The proof of this in a way is quite simple: if $f$ exists, it should be given by $\sum_{i=0}^{\infty} \alpha(a, a^{i-1}x)$ or $\sum_{i=-\infty}^{-1} \alpha(a, a^{i-1}x)$ if these formal expressions make any sense. In general they don’t. But if we know exponential mixing for $a$, then these sums exist as distributions on Hölder functions. Using exponential mixing together with the so-called higher rank trick, one argues that both the sum over the past and the future are the same. Furthermore, differentiating these expressions for $a$ close to $\ker \lambda$ for the Lyapunov exponents $\lambda$ will still converge as distributions dual to Hölder functions. From this one can use basic results in PDE to show they are $C^\infty$.

In a related work, we show similar results for certain partially hyperbolic actions [8], essentially implementing ideas from hypoelliptic operators and especially Hörmander’s square theorem. The main new idea here to use more sophisticated arguments form PDE to prove global smoothness if one has smoothness of a function along smooth distributions which generate the tangent bundle. This basic argument has been used repeatedly, and is now quite polished and simple. Rauch has generalized these results to distributions with partial derivatives of all orders that belong to a fixed Sobolev space [91].
This argument has been used extensively over the last two decades. In particular a variation of it allowed for the global rigidity results for higher rank Anosov actions on tori and nilmanifolds \cite{92,57}. That these actions have exponential mixing for Hölder functions has also been very successfully used in homogeneous dynamics. For the case of semisimple Lie groups it follows from representation theory. For tori, it is fairly basic Fourier analysis. For nilmanifolds, it follows from an application by Gorodnik and myself \cite{62} of polynomial equidistribution on nilmanifolds by Greene and Tao \cite{63}. In addition, the core of the argument is also present in the classification of Anosov $\mathbb{Z}^k$ actions on tori and nilmanifolds \cite{92,57}. Indeed, we are essentially smoothing a twisted cocycle.

With Schmidt, Katok obtained rigidity results for Hölder cocycles of higher rank expansive and mixing actions by automorphisms of compact abelian groups. While the basic ideas are related to the case of manifolds, the structure theory of automorphisms of compact groups is highly complicated and requires extensive adjustments \cite{10}.

Back to the partially hyperbolic smooth actions, Katok and Kononenko (in his thesis) developed a rather different approach, using periodic cycle functionals \cite{13}. For rank 1 Anosov actions, the driving force behind cohomology vanishing is the Livsic’ theorem. An obvious obstruction for a function $\phi$ to be a coboundary is that $\phi$ sums over periodic orbits are 0. In the case of a partially hyperbolic diffeomorphism $f$ or flow, there may however not be any periodic orbits. Instead, suppose you have a finite sequence of points $x_k$ connected by either stable or unstable manifolds of $f$, called a periodic cycle $C$. Then define $P^*(x_k, x_{k+1}) = \sum_{l=0}^{\infty} \phi(f^l(x_{k+1}) - \phi(f^l(x_k))$, if $x_{k+1}$ and $x_k$ belong to the same stable manifold, and similarly when subsequent points lie in an unstable manifold, summing over the past in this case. In either case, the sums will converge thanks to the exponential decay in the distances and the $\phi$ being Hölder. Then define the periodic cycle functional attached to $C$ as the sum $P(C) = \sum_k P^*(x_k, x_{k+1})$. Call $f$ locally accessible if any two nearby points can be connected via finitely many stable and unstable manifolds locally. They then show:

For locally accessible $f$ a Hölder function $\phi$ is a coboundary of a Hölder function precisely when the periodic cycle functionals vanish.

The idea is that we can define a solution from one point to the next if they lie on the same stable or unstable manifold. The periodic cycles condition then guarantees that the solution is (locally) well-defined.

Katok and Kononenko then explain a variety of situations when this technology applies, in particular for contact Anosov flows. Wilkinson in \cite{100} extends this work to accessible partially hyperbolic systems. These results pave the way for studying local rigidity and stability properties for partially hyperbolic flows and diffeomorphisms we will address below. Ultimately studying cycle relations play a major role in \cite{95} where the Katok Spatzier Conjecture is solved under fairly general conditions for totally Cartan actions.
Closely related is Katok’s paper with Nitica and Török in [18]. There they devise a geometric approach to smooth cohomology vanishing of higher rank Anosov \( Z^k \) or \( R^k \) actions valid for “totally non-symplectic” (TNS) actions and “small” cocycles. The second condition simply means that the cocycle takes values close enough to the identity in the target group - \( R \) or more generally a Lie group in this paper. The TNS condition means that no two Lyapunov exponents, viewed as linear functionals on \( R^k \) are negatively proportional. The main consequence is that any two Oseledets spaces can be put inside a stable manifold of some element. Single Anosov diffeomorphisms or flows are of course never TNS. The coboundary is defined similarly to the work with Kononenko. The TNS condition is used to prove smoothness of the solution. The point is that the solution is smooth along stable manifolds, hence along any pair of the coarse Lyapunov spaces (tangent to the Oseledets spaces) - thanks to the TNS condition. Similar, more complicated arguments apply to small cocycles with values in a Lie group. They also get results for some Weyl chamber flows using an approach via differential forms. Nitica and Török apply these ideas to cocycles with values in diffeomorphism groups, naturally a very technical project [88, 89].

Finally, let us discuss Katok’s work with Damjanović on restrictions of Weyl chamber flows to generic higher rank subgroups of the \( R^k \), the Cartan subgroup from [20]. Here they call a subgroup generic if it intersects the kernel of two Lyapunov exponents in different lines.

The action of any generic subspaces of the diagonal subgroup of \( SL(n,R) \) is cocycle rigid both in Hölder and \( C^\infty \) categories.

Essentially they use the periodic cycles functionals from Katok and Kononenko. As discussed there, all is well in the TNS case when there are no negatively proportional Lyapunov functionals, and one can put any pair of coarse Lyapunov foliations inside one stable manifold. In the semisimple case however, \( SL(n,R) \) for example, one needs to commute negatively proportional one-parameter subgroups (opposite horocyclic flows). This leads them to study the group of relations between unipotent generators called the Steinberg group. Composing with a periodic cycle functions yields a continuous relation (Steinberg symbol). For \( SL(n,R) \), the latter are finite, and define a homomorphism from the fundamental group \( \Gamma \) of the space to \( R \). By Kazhdan’s property of \( \Gamma \), the latter has to be trivial, as desired.

Katok and Nitica pursued these ideas for cocycles over partially hyperbolic systems taking values in diffeomorphism groups and are close to the identity on generators [21]. This is natural and interesting as perturbations give rise to small cocycles.

For actions of abelian groups, one can also define higher cohomology groups of the action, either via a natural chain complex or - in the case of \( R^k \) actions - in terms of differential forms tangent to the orbit foliation. This was discussed extensively by A. Katok and S. Katok in their papers [9, 19]:
Consider an action of \( \mathbb{Z}^k \) by toral automorphisms, in either the Anosov or the partially hyperbolic setting. Then higher cohomologies vanish up to the rank \( k \). For the \( k \)'th cohomology, there are countable many periodic orbit obstructions to vanishing.

Essentially they extend the Livisic’ theorem about periodic data obstructions to \( C^\infty \) \( k \)-cocycles, and the vanishing of 1-cohomology in higher rank à la Katok-Spatzier to the vanishing of the \( l \)-cohomology for \( l < k \). The arguments proceed via finding solutions in terms of Fourier series, and controlling the dual orbits carefully to give the desired regularity.

8. Normal Forms

A normal form is a coordinate system in which a diffeomorphism or flow takes on polynomial form. Classically, they were used to describe an ODE near an equilibrium or fixed point, and the history goes back to Poincaré. For hyperbolic fixed points, Hartman and Grobman gave a continuous linearization which Sternberg improved under non-resonance conditions. In general one gets coordinates with respect to which the map has polynomial form. To understand local rigidity issues, it became important to develop normal forms theory along whole orbits of an action, i.e. a non-stationary version. This first emerged in Katok’s work with Lewis on local rigidity of the \( SL(n, \mathbb{Z}) \) action on the \( n \)-torus \( T^n \) where they linearized maps along one-dimensional contracting foliations [6]. With Guysinsky, this developed into a general theory of normal forms along contracting foliations \( \mathcal{F} \) [17].

There are coordinate charts for each leaf of \( \mathcal{F} \), depending Hölder on the base point, such that the map restricted to the leaves of \( \mathcal{F} \) written in the respective coordinates are polynomial.

Much more, they could control the degree of the polynomial and even find a group of fixed degree polynomials that these chart maps belong to. They had to pay a price though and require that the Mather spectrum of the derivative acting on vector fields is contained in suitably narrow bands. Fortunately this condition always holds in algebraic settings, and also in perturbations of such.

Katok and Spatzier used these normal forms in their main work on local rigidity [16]. We crucially used that the normal forms of a given contracting map will also be normal forms for commuting maps. This immediately gave normal forms for the whole \( \mathbb{R}^k \) or \( \mathbb{Z}^k \) action along contracting foliations such as the coarse Lyapunov foliations. As mentioned earlier, one key argument involved studying returns of the leaf of a coarse Laypunov foliation to itself under the maps which act isometrically along the given coarse Lyapunov foliation. Having these maps take polynomial from of fixed degree in a finite dimensional Lie group forced \( C^0 \) convergence to become \( C^\infty \) convergence.

Non-stationary normal forms theory were developed considerably since, especially in the works of Kalinin and Sadovskaya under various conditions like
pinching or presence of a quasi-conformal structure. Kalinin wrote the definitive version of the theory in the uniformly hyperbolic case in [75]. We refer there for more history as well.

Finally, there are even measurable normal forms for a measure preserving diffeomorphism, i.e. in the context of Pesin theory. Then Kalinin and Sadovskaya as well as Melnick find normal forms on stable manifolds [78, 86] using very different methods. Kalinin and Sadovskaya carefully extend the uniformly hyperbolic methods while Melnick uses a more differential geometric approach providing invariant higher order structures that are preserved. In the strict $\frac{1}{2}$-pinching case, there are just invariant affine connections which makes this very appealing. Then the exponential map serves as normal coordinate chart. In either case, thanks to the existence of Lyapunov exponents the Mather spectral criterion is easily satisfied and no assumptions need to be made. While in principle much weaker than Hölder normal forms, these measurable normal forms have been applied to prove rigidity results, in [56] to classify higher rank abelian totally non-symplectic actions on tori and nilmanifolds and in [48] to classify higher hyperbolic rank Riemannian metrics with $\frac{1}{4}$-pinched sectional curvature $K$. These are manifolds with $-\frac{1}{4} \geq K \geq -1$ such that every geodesic $c(t)$ has a perpendicular parallel field that makes curvature $-1$ with $c'(t)$ for all $t \in \mathbb{R}$.

9. PARTIALLY HYPERBOLIC ACTIONS, K-THEORY AND LOCAL RIGIDITY

After proving local rigidity of the standard higher rank abelian actions, pursuing local rigidity for partially hyperbolic algebraic actions (by automorphisms or left translations on homogeneous spaces $G/\Gamma$) is more than natural. This is not just a technical exercise. One main problem is the following: for Anosov actions, thanks to structural stability, there is a conjugacy (or at least orbit equivalence) between perturbed and original actions. This is missing. Worse, in the Anosov case, we used exponential mixing of the perturbed action to straighten out time. In general we do not know if exponential mixing is a stable property, and in fact used the Hölder conjugacy to get it from exponential mixing of the algebraic action. New tools were clearly needed.

Again with Damjanović, Katok extends the ideas about cocycle rigidity for $SL(n, \mathbb{R})/\Gamma$ to local rigidity in [24]. This is a geometric approach, it does not use exponential mixing, and can be made to work for perturbations:

*Weyl chamber flows restricted to generic higher rank subgroups of the Cartan subgroup are $C^{1,\infty}$ locally rigid.*

The proofs are much inspired by the earlier work on cocycles, and crucially use the Steinberg relations or equivalently $K$-theory. However knowledge of the $K$ theory of a general real semisimple Lie group is limited, the calculations daunting. Z. Wang generalized these ideas to more groups in her thesis with Katok. Then another Katok student, Vinhage, had a surprising new idea in his thesis: he approached this via actions of free products of suitable Lie groups - some a priori horrible infinite dimensional topological group - and a basic result of
Gleason and Palais from transformation groups that characterizes when an action of a connected locally connected topological group forces the group to be a Lie group \[98\]. After the fact, the Lie groups in this free product correspond to unipotent groups acting along coarse Lyapunov foliations. Eventually, Vinhage and Wang joined forces and proved a rather general version of local rigidity of partially hyperbolic algebraic actions under an accessibility condition \[99\]. The work by Spatzier and Vinhage in \[95\] crucially uses the free products approach to classify totally Cartan actions on a general closed manifold.

10. **Partially Hyperbolic Actions, KAM and Local Rigidity**

In our previous discussions of partially hyperbolic actions, it was crucial that distinct points were accessible via finitely many paths of stable and unstable manifolds. Except for Anosov actions, this is of course never true for partially hyperbolic actions by automorphisms on the torus as stable and unstable spaces are just linear subspaces. So a different playbook is needed. The classical tool to study perturbations of isometries and completely integrable systems is KAM theory. One tries to find a solution to an equation by finding an almost solution and then use this to find a closer one. If the convergence is fast enough one can get an actual solution as a limiting map. This idea had not only been used to great effect in studying classical Hamiltonian systems but also for local rigidity in the Zimmer program by Fisher and Margulis \[55\] for understanding perturbations of isometric actions of property (T) groups.

Damjanović and Katok use a KAM approach to prove in \[23\]:

*Higher rank abelian groups acting on tori by automorphisms are \(C^{k,\infty}\)-locally rigid for some predetermined \(k\).*

Unlike in classical KAM theory, they do not require explicit Diophantine conditions but instead need a higher rank action. The scheme is much like the classical Newton method: given an approximate solution you improve upon it by solving a linearized equation at the approximate solution. Solving it yields a new approximate solution quadratically closer then the original one. Here commutativity of the action is used explicitly via a variation of the “higher rank trick” from \[7\]: if small perturbations of two linear automorphisms commute then certain obstructions to solving the linearized equation almost vanish. This then allows to make the error terms of the next step in the KAM scheme quadratically closer. All along there is finite loss of derivatives which determines \(k\) in the main result.

Damjanović and Katok further study local rigidity of commuting unipotent one-parameter subgroups on a homogeneous space \(G/\Gamma\) in a semisimple group \(G\) with uniform lattice \(\Gamma\) \[26\]. Now there are deformations, already in the homogeneous category. Essentially they show that these are the only ones, at least for particular cases like \(G = SL(2, \mathbb{R})\) or \(G = SL(2, \mathbb{C})\). This followed earlier rigidity results for cocycles over such actions by Mieczkowski and Ramirez \[87, 90\]. Later, Z. Wang obtained many results in this direction (cf. e.g. \[97\]).
11. CONCLUSION

It is amazing to see the web of connections between the various problems in the study of geometric rigidity, how pieces fairly far apart and tools developed for very different purposes became important in other parts of the program. Katok saw many of these ideas and their possibilities early on, and was instrumental in developing a good part of them. Moreover, the connections between geometric rigidity and measure rigidity are close, and the impact of the ideas on homogeneous dynamics and number theory cannot be underestimated.

Katok was also deeply involved in the training of many graduate and undergraduate students, and postdocs, and in organizing many conferences and workshops worldwide. I certainly benefitted greatly from these opportunities as did many others.

REFERENCES

In this volume


In volume 1


Other sources


19


[90] F. Ramirez, Cocycles over higher-rank abelian actions on quotients of semisimple Lie groups, *J. Mod. Dyn.*, **3** (2009), 335–357.


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