Semiregular Tilings

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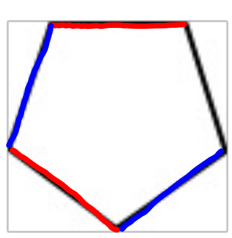
Recall: A tiling is semiregular if it is made of 2 or more kinds of regular polygons, and the arrangement of polygons around any vertex is the same as any other.

We want a complete list of all semiregular tilings. We will start by making a list of combinations of regular polygons that have angles that add up to $360°$, as this is necessary for a tiling to be possible. Then we will show the combinations that can actually be used for a semiregular tiling. First, however, we will note an important observation that helps rule out some possibilities, which will reduce our list of possibilities.

Observation: There are no semiregular tilings of the form $a.b.c$, where a is odd and $b \neq c$. (note that $a=b$ is included in the set of impossible tilings.)

Proof Since there are only 3 polygons, two adjacent sides of any a-gon must be different kinds of polygons, that is, one is a b-gon, and one a c-gon. So if we color each edge of an a-gon red if it is adjacent to a b-gon and blue if it is adjacent to a c-gon, then the edges will have to alternate between blue and red. But this is impossible when a is odd.



← Cannot be either red or blue.

Possible combinations. We know regular n-gons must have angles of $\left(\frac{360(n-2)}{n}\right)^\circ$. This gives:

sides	3	4	5	6	7	8	9	10	11	12
angle measure (degrees)	60	90	108	120	$128\frac{4}{7}$	135	140	144	$147\frac{3}{8}$	150

We will organize the list of possibilities first by the number of triangles.

- The most triangles we could have is 4, since with 5, the only polygon with small enough angles to fit is another triangle. Four triangles leaves $360 - 4 \cdot 60 = 120^\circ$ to be filled. One hexagon is the only combination of 4- or more sided polygons that will fit. 3. 3. 3. 3. 6

- With 3 triangles, we have $360 - 3 \cdot 60 = 180^\circ$ left to fill. We can only do this with 2 squares. 3. 3. 3. 4. 4 or 3. 3. 4. 3. 4

  (These are the only two options: the squares can be adjacent or separated.)

- With 2 triangles, $360 - 120 = 240^\circ$ remain. Two squares would leave room only for another triangle. If we use 1 square, we get 150° remaining. We can't fill this with two polygons with 5 or more sides, and the observation tells us 3.3.4.12 won't work.

  If there are no squares, there is room for 2 hexagons only. 3. 3. 6. 6 or 3. 6. 3. 6

- With 1 triangle, 300° remain. 3 squares is impossible. With 2 squares, we have exactly enough room for 1 hexagon. 3. 4. 4. 6, 3. 4. 6. 4

  With 1 square, 210° remain, which cannot be filled by any 5- or larger gons.

If there are no squares, then there is only room for 2 other polygons. But our observation tells us this can only happen if they have the same number of sides, so the only possibility is 3.12.12.

- With no triangles, we'll now think about squares. 2 squares leaves 180°, which is too big for any 1 polygon and too small for 2 polygons when triangles are not allowed. One square leaves room for 1 hexagon and one 12-gon.

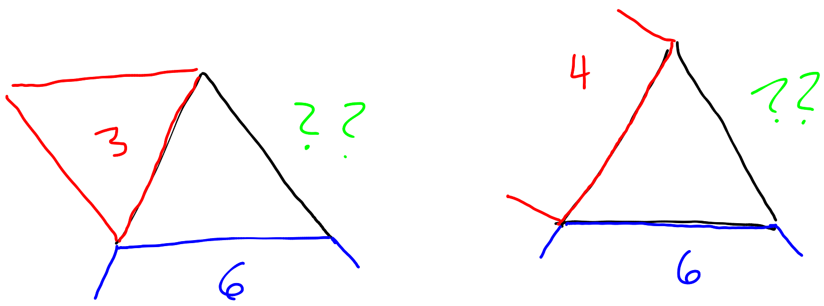
4.6.12

one square and one pentagon or 7-gon leaves room that can't be filled. We can do one square and 2 8-gons. One square and anything larger doesn't leave room for anything else.

4.8.8

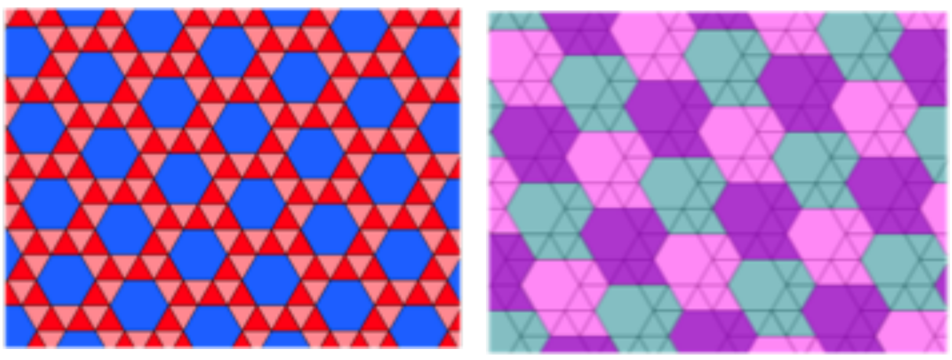
- With no squares, if we had a pentagon, there are no two polygons that will fill the remaining $360-108=252°$. If all polygons have at least 6 sides, and one has at least 7, then the sum of the angles must be greater than 360°.

Now we have a list of ten possibilities. Two of these are impossible, for essentially the same reason as the observation: 3.3.6.6 and 3.4.4.6. Given any pair of sides of the triangle, they must be adjacent to polygons with different numbers of sides. But this cannot happen for all 3 pairs of sides.

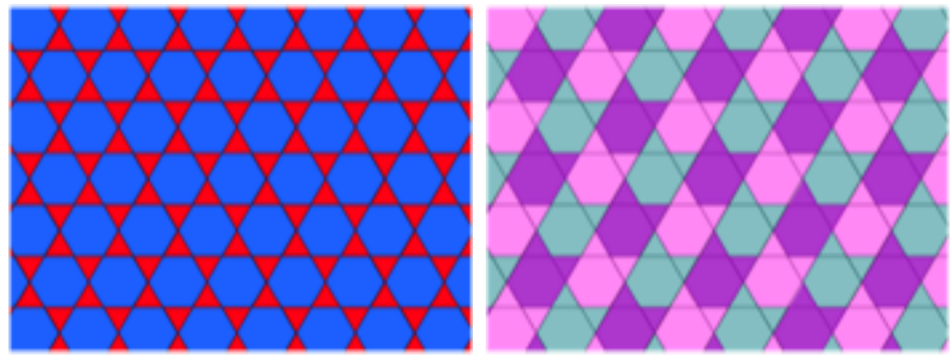


Next we will show that the 8 remaining possibilities can all be realized.

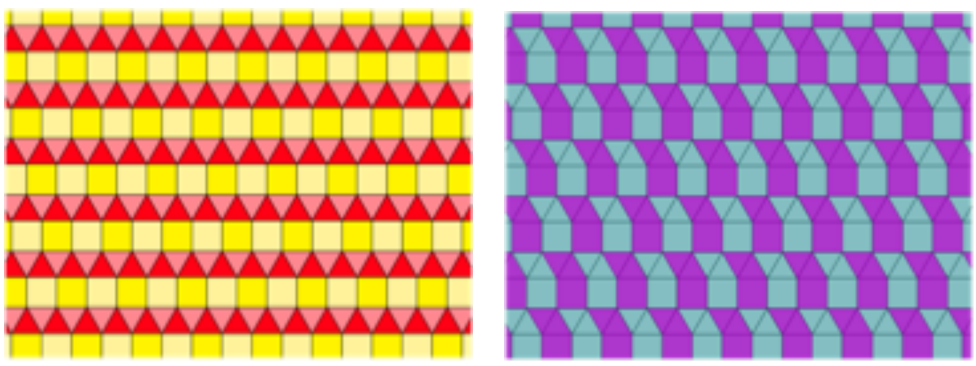
We will show each tiling in two ways: one that shows the polygons clearly, and one that shows a fundamental domain, that is, a single shape that can be repeated to give the tiling.



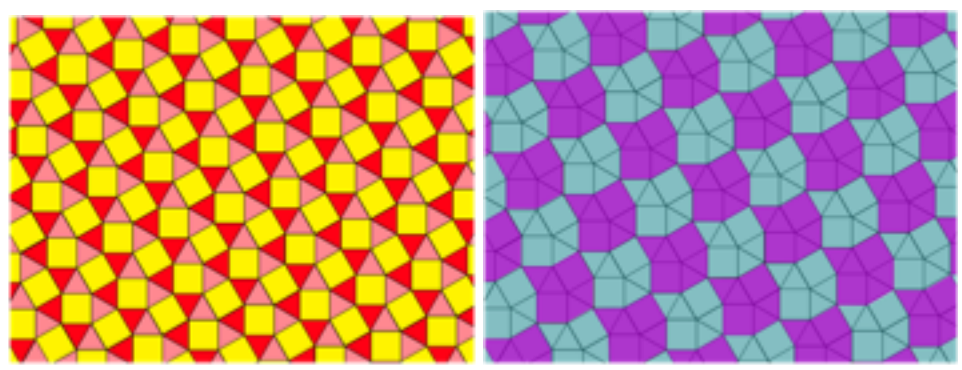
3.3.3.3.6



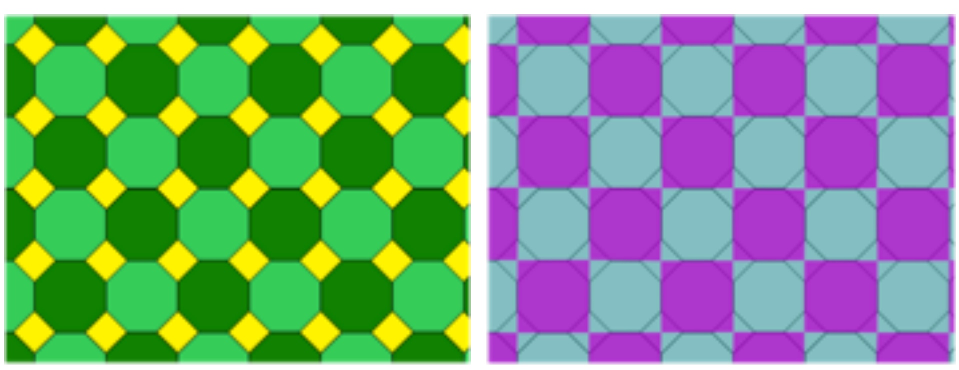
3.6.3.6



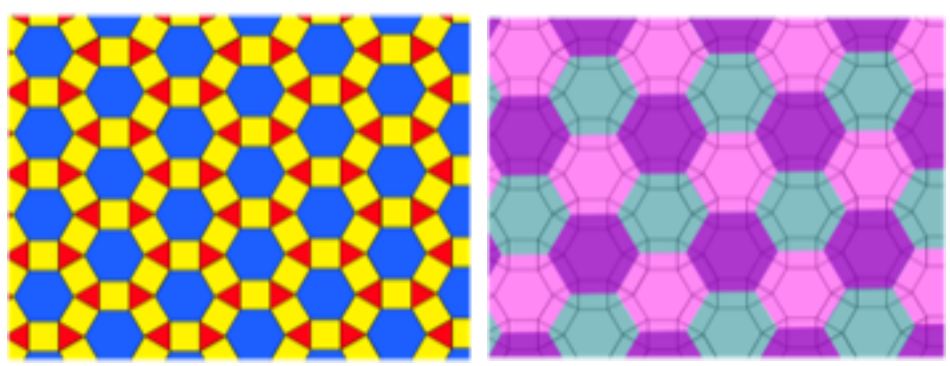
3.3.3.4.4



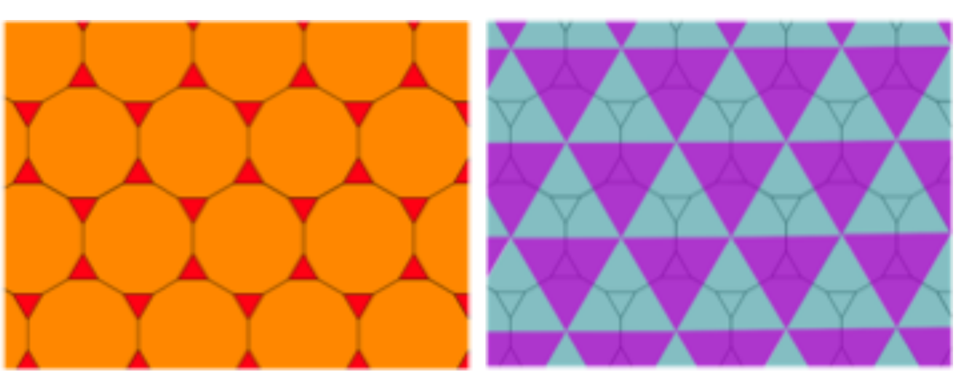
3.3.4.3.4



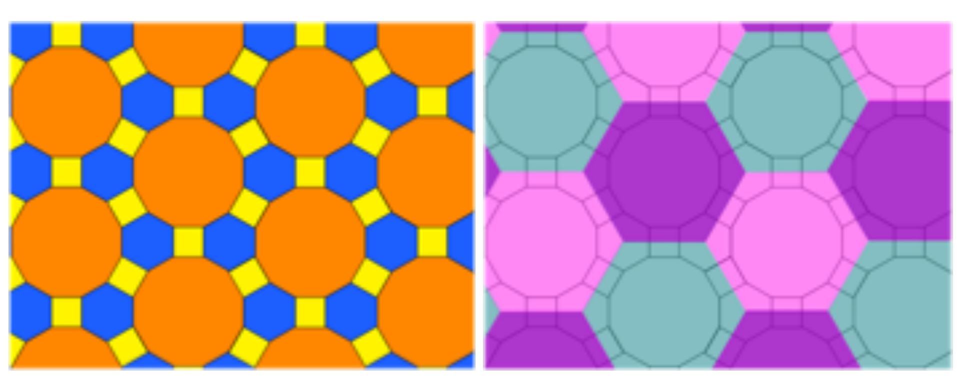
4.8.8



3.4.6.4



3.12.12



4.6.12